Tight smoothing of the squared distance functions and applications to computer-aided design and differential equations.

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Tight smoothing of the squared distance functions and applications to computer-aided design and differential equations

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Abstract

We study the quadratic lower compensated convex transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to a nonempty, non-convex closed set $K \subset \mathbb{R}^n$. These transforms, introduced in [20], provide $C^{1,1}$-smooth tight-approximations of the squared distance functions. We introduce a result to calculate explicit formulae for the lower transform of the squared distance functions. Our first main result is the geometric characterisation of critical points of the quadratic lower compensated convex transform $C^l_\lambda \mathrm{dist}^2(x,K)$, which shows that the geometric non-smooth critical points of the squared distance function are identical to the usual critical points for the smoothed squared distance function. We then consider $K \subset \mathbb{R}^n$ be finite and classify the Morse indices of critical points of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ in $\mathbb{R}^2$ and $\mathbb{R}^3$, that is, classify critical points into non-degenerate critical points and degenerate critical points. This classification of Morse indices of critical points cannot be fully justified without knowing the behaviour of the lower transform of the squared distance function to finite sets. Therefore, we study some local properties of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ for the squared distance functions to finite sets to understand its behaviour in a small neighbourhood of critical points. We show that the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ has a semi-global property in $\mathbb{R}^2$, that is, for $\lambda > 0$ sufficiently large, the lower transform of the squared distance function equals the squared distance function except on a small subset of a given bounded set. Under certain regularity assumptions of K, we establish that the lower transform of the squared distance function to finite sets has a semi-global representation in $C(K)$ under triangulations, which helps us to understand the role of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ in surface reconstruction.

Then we calculate some explicit formulae of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ for finite sets and their corresponding first order and second order differential equations. These are helpful in applications of the lower transform of the squared distance functions which exploit the crucial tight approximation property of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$.

One application of the lower transform of the squared distance function to finite sets is to surface reconstruction. We modify an approach by T. K. Dey [6], that uses only the stable manifolds of non-degenerate geometric critical points of the squared distance functions to finite sets to reconstruct surfaces by using instead the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$, and illustrate with examples that stable manifolds of both non-degenerate and degenerate critical points can be used in surface reconstruction. We also show that the second order smooth gradient system of the lower transform of the squared distance function enjoys unique global solution, in contrast to the original non-smooth systems. Another application is to the systems of second order differential equations; we use second order smooth gradient systems to investigate the effects of smoothing on the long-time dynamics of a physics model of Y. Brenier [5] governed by half of the squared distance function to set $K = \{-1, 1\}$. We also use a qualitative approach to study the long-time dynamics of second order smooth gradient system of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to set $K = \{-1, 1\} \subset \mathbb{R}$ and we study the behaviour of solutions when $\lambda \to \infty$.

Chapter 1

Introduction

1.1 Background and Motivations

The notions of quadratic compensated convex transforms were first introduced by Kewei Zhang [20]. The lower quadratic compensated convex transforms have been used in the study of quasiconvex relaxation and gradient of Young measures in the calculus of variations [22, 24, 25, 26]. The lower and upper quadratic compensated convex transforms $C^l_{2,\lambda}(f)$ and $C^u_{2,\lambda}(f)$, defined by 2.2.10 and 2.2.11 in Chapter 2, (for short, lower transform $C^l_\lambda(f)$ and upper transform $C^u_\lambda(f)$) for a given function $f : \mathbb{R}^n \to \mathbb{R}$, for both small and large $\lambda > 0$, were discussed in [20], which will be very useful in our current research on the lower transforms. The lower transform $C^l_\lambda(f(x))$ for function $f \in C^{1,1}_{loc}$ attains the value $f(x)$ for a finite value of λ, which depends on x. Such a property of the lower transform is called a 'tight' approximation of $C^l_\lambda(f)$ to the original function f as $\lambda \to +\infty$. This tight approximation property and some other properties for the lower transform $C^l_\lambda(f)$ for a lower semi continuous function f that maps bounded sets to bounded sets were established by Kewei Zhange [20, Recovery/Approximation Theorem 2.3] which we also have included as Theorem 2.2.15 in Chapter 2.

The lower transform $C^l_\lambda(f)$ of a function f bounded below is considered, geometrically, as a better smooth approximation than the convex envelope $C(f)$ and other approximations because $C^l_\lambda(f)$ is a tight approximation from below. We also note that it has been shown in [1, Theorem 2.5] that the quadratic transforms are tighter approximations to the original functions than the Moreau envelopes. It was illustrated with help of example [1, Example 5.4] that the approximation and smoothing effects of the quadratic transforms upon the original functions for large parameters are more predictable than the Moreau envelopes and Lasry-Lions regularization [20, see section 5].

We are mainly interested in the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the Euclidean squared distance function $\mathrm{dist}^2(x, K)$ to a closed set $K \subset \mathbb{R}^n$ and its analytic and geometric properties. Some properties had previously been studied, including that the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance functions to the closed set K are $C^{1,1}$ approximations of the original squared distance functions for $\lambda > 0$ in [20, 21]. The tight smoothing of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ and its many applications in different areas of mathematics [11, 22, 23, 24, 25, 26, 27] and computer science [3, 4, 15, 19] motivated us to study this smooth approximation of the squared distance function.

We write 2-dimensional numerical schemes for the lower, upper and mixed transforms with the help of Orlando Antonio (see Appendix C) using Adam M. Oberman's monotone scheme. Oberman [14], first introduced this monotone scheme for computing the convex envelope $C(f)$ of a given function $f : \mathbb{R}^n \to \mathbb{R}$. Therefore, using this scheme to compute convex envelope gives us an easy way to compute the transforms, which are better approximation. The implementation of this scheme through MATLAB programming can calculate and plot, a given function, its corresponding convex envelope, lower transform $C^l_\lambda(f)$, upper transform $C^u_\mu(f)$ and mixed transform $C^l_\lambda C^u_\mu(f)$ for parameters $\lambda > 0$ and $\mu > 0$; the details can be seen in the Appendix C. By applying the lower transform to the squared distance functions, we clearly see the smoothing made by the lower transform $C^l_\lambda(\mathrm{dist}^2(\cdot, K))$ to the original squared distance function $\mathrm{dist}^2(\cdot, K)$ for a finite set $K \subset \mathbb{R}^n$ for $n = 1, 2$ from the results of plotting 4.2, 4.3 and 4.4 obtained from MATLAB implementation and using Mathematica in Chapter 4.

Our first main result (Theorem 3.1.1), is a property of the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the Euclidean squared distance functions to closed set $K \subset \mathbb{R}^n$ which other types of smoothing might not have. This result shows that non-smooth geometric critical points of the squared distance function and the usual critical points of the smoothed squared distance functions are the same. We study local properties of the lower transforms $C^l_\lambda \mathrm{dist}^2(x,K)$ and introduce a semi-global result (Theorem 4.1.6), which are used to justify the classification of the Morse indices of critical points in $\mathbb{R}^2$ and $\mathbb{R}^3$, which are required to understand stable manifold of critical points. To understand the application of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ in global triangulation simplices, we provide the result (Theorem 4.1.7), about the triangulation simplices of finite set K.

We study the properties of the lower transforms $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance functions to a closed set $K \subset \mathbb{R}^n$ and apply them to surface reconstruction problem [6]. The problem of reconstructing the surfaces of three-dimensional objects has diffenent approaches. Some previous approaches to the surface reconstruction problem would assume additional structure along the given data points, such as, the construction of three-dimensional surface from a set of planar contours [12] and surface construction over a set of cross-sectional contours [8]. The general surface reconstruction problem where no additional structure is given along with a finite set of points in $\mathbb{R}^3$ was considered in [7] and [6]. The applications of the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot,K)$ to the approach by T. K. Dey [6] of surface reconstruction in $\mathbb{R}^3$ is one of the motivations for current research.

Dey used the stable manifolds of index 2 non-degenerate critical points for surface reconstruction and did not discuss the surface reconstruction using the stable manifolds of degenerate critical points. For the reconstruction purposes, he decomposed $\mathbb{R}^3$ using the stable manifolds (i.e., the simplicial complexes). On the other hand, in our approach using a tight approximation (lower transform) of the squared distance function to finite sets, we consider both non-degenerate and degenerate critical points for the surface reconstruction, which is an advantage over the approach of Dey. Further, since the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot,K)$ is $C^{1,1}$ [20, Theorem 3.1] while the squared distance function is not continuously differentiable, our approach has a well-defined gradient at every point $\mathbb{R}^n$ and the gradient systems of the lower transform for the squared distance functions to finite sets enjoy unique global solutions, in contrast to the original non-smooth systems.

Furthermore, in the approach of Dey [6] for surface reconstruction using the squared distance function to finite sets in $\mathbb{R}^3$, he cannot use the theory of differential equation as the squared distance function has points where the gradient does not exist, and thus he has to work hard to define flow curves and associated stable manifolds. On the other hand, the advantage of the lower transform over the squared distance function is that, the gradient of the smooth approximation $C^l_\lambda \text{dist}^2(x, K)$ of the squared distance function to the finite set $K \subset \mathbb{R}^3$ can be easily calculated and will still be pointing in the same direction as that shown in [6] whenever $C^l_\lambda \text{dist}^2(x, K) = \text{dist}^2(x, K)$, which holds except on a small subset of a given bounded set when λ is sufficiently large and is shown in Chapter 4, Theorem 4.1.6. and thus has advantages over the squared distance function to finite sets in $\mathbb{R}^3$.

Another application is to second order differential equations; we use second order smooth gradient systems to investigate the effects of smoothing on the longtime dynamics of a physics model of Y. Brenier [5]. Note that the solutions of the second order gradient system of the lower transform $C^l_\lambda \text{dist}^2(x, K)$ for the squared distance function to a finite set $K \subset \mathbb{R}^n$ also enjoy unique global solutions. We compare the solution of Yann Brenier's dynamical system (see [5, Expression (2.19)]), governed by half of the squared distance function $\Phi := \inf \left\{ \frac{|x-s|^2}{2};\ s \in K \right\} = \frac{1}{2}\text{dist}^2(x, K)$, that is,

$$\frac{d^2x}{dt^2} = D\Phi(x),$$

where D denotes the gradient operator and $K = \{-1, 1\}$, with that of the lower transform $C^l_\lambda \text{dist}^2(x, K)$. This will be discussed in Chapter 6, that the long-time behaviour of the solution of second order gradient system of the lower transform $C^l_\lambda \text{dist}^2(x, K)$ to set $K = \{-1, 1\}$ is more predictable than the solution of Brenier dynamical system[5]. We will also use explicitly calculated solutions of the second order graident systems of one and two-dimensional examples of the lower transforms $C^l_\lambda \text{dist}^2(x, K)$ to set $K = \{-1, 1\}$ and $K = \{(-1, 0), (1, 0)\}$ and understand the long-time behaviour of the solutions, in fact, we will show the oscillating behaviour of solutions when the initial data are restricted. The qualitative behvaiour of the solutions of these systems of the lower transform $\frac{1}{2}C^l_\lambda \text{dist}^2(x, K)$ to set $K = \{-1, 1\}$ based on a conservation law (see 6.0.1 Chapter 6) help us to understand the long-time behaviour. We also study the behaviour of solutions of the modified system when $\lambda \to \infty$.

1.2 Thesis Overview

The layout of this thesis have been arranged as Chapters. The first chapter Introduction is followed by the Chapter 2 Preliminaries where we will introduce some basic definitions, facts and related Theorems and Lemmas. Most of the preliminaries are taken from different books and research publications provided with clear references.

- In Chapter 3, we prove that the non-smooth geometric critical points of the squared distance function are the same as the usual critical points of the smoothed squared distance function. Then we illustrate the Morse indices of non-degenerate critical points and degenerate critical points with the help of propotype examples. Finally, we classify the Morse indices of the critical points of $C^l_\lambda \mathrm{dist}^2(x, K)$ for finite sets K in $\mathbb{R}^2$ and $\mathbb{R}^3$, which are some of our interesting new results.

- In Chapter 4 , we show how the lower transform modifies the original squared distance function near the medial axis when $\lambda > 0$ is sufficiently large, which gives information about how the gradient flow will move near the singular points of the squared distance function. It also includes locality results of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to finite set $K \subset \mathbb{R}^n$ and semi-global Theorem 4.1.6. We establish a result of representation of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ in each triangulated simplex of the finite set K under certain conditions. We prove a global result for triangulated simplices which shows that for simplices in the triangulation of the set K, the lower transform is the same as that for a single simplex in the collection of triangulation under certain regularity assumptions on K. We also calculate some explicit formulae for the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ for the squared distance function to finite sets K in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, that can be used in practical applications. These examples show the effects of the smooth approximation of squared distance functions to the original squared distance functions.

- In Chapter 5, we discuss the application of the lower transform of squared distance function to finite sets to surface reconstruction. An approach by T. K. Dey [6] using non-smooth function for surface reconstruction is modified and instead of using the squared distance function to finite sets we used the lower transform of the squared distance functions to finite sets. Since $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ is $C^{1,1}$, the associated dynamical system is smooth. Therefore, our reconstruction approach is more automatic than that of Dey. We mainly show, in general as well as illustrating with examples, that the surface reconstruction can be done by non-degenerate critical points and degenerate critical points of the lower transform of the squared distance functions to finite sets.

- Chapter 6 introduces the applications of the lower transform of squared distance functions to finite sets to ordinary differential equations and show the existence of unique global solution for such a transform. Further, we compare smoothing effects of the lower transforms $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ with the Y. Brenier smoothing on dynamical systems [5]. We investigate the long-time behaviour of the solution of second order gradient system of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ to set $K = \{-1, 1\}$ and show the solution oscillates for certain initial data. In addition, we also give a qualitative discussion of this example of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ to set $K = \{-1, 1\}$, which gives more information about the behaviour of the solution.

- Chapter 7 presents the detailed computational work done that is helpful throughout this document. It is mainly on first and second order gradient systems of particular examples of the lower transform of squared distance functions of different finite sets of Chapter 4 and the solutions of these smooth gradient systems. Finally, we will conclude with our future direction of work and goals. In the Appendix we included the extra calculations and simplifications, and the MATLAB programming code for calculating and plotting of the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$, upper transform $C^u_\lambda(f)$ and mixed transforms.

Chapter 2

Preliminaries

2.1 Basics

We will try to make the thesis self-contained, so we shall list in this chapter all the well known related concepts. We also provide the results that will be needed in establishing our new results. We make precise notations that will be used throughout the document unless stated otherwise. We denote the Euclidean space $\mathbb{R}^n$ the vector space of all n-vectors in general and in particular, the Euclidean space $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. The non-empty, non-convex closed set K is subset of $\mathbb{R}^n$ and M_K (see Definition 2.1.3) denotes the medial axis of K. We denote the convex hull of the set K by $C(K)$ and its relative interior of set K by $ri\ C(K)$. The gradient operator in $\mathbb{R}^n$ is denoted by D (note that other authors sometimes use ∇ as the gradient operator). We denote a circle with radius $r > 0$ centered at $(0,0)$ by $\mathbb{S}^1\big((0,0),r\big)$ and sphere centered at $(0,0,0)$ by $\mathbb{S}^2\big((0,0,0),r\big)$ defined as

$$\begin{aligned} \mathbb{S}^1\big((0,0),r\big) &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\} \\ \mathbb{S}^2\big((0,0,0),r\big) &= \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}. \end{aligned}$$

For more details about the preliminaries one can read [6, 14, 17, 18] and the work on compensated convexity in [20].

Definition 2.1.1. Let K be a non-empty, non-convex and closed subset of $\mathbb{R}^n$, then for every $x \in \mathbb{R}^n$ the Euclidean distance from x to K is defined by $\operatorname{dist}(x, K) := \min_{y \in K} |x - y|$, and hence the **Euclidean squared distance** is

$$\operatorname{dist}^2(x, K) := \min_{y \in K} |x - y|^2.$$

Definition 2.1.2. For a point $x \in \mathbb{R}^n$ the **open ball** $B(x, r)$ centred at x with radius $r > 0$ is defined by

$$B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}.$$

The **closed ball** denoted by $\bar{B}(x, r)$ centred at x with radius $r > 0$ is defined as

$$\bar{B}(x, r) := \{y \in \mathbb{R}^n : |x - y| \leq r\}.$$

In $\mathbb{R}^2$ the closed ball is called closed disc. A ball $B(0, 1)$ centred at $0 \in \mathbb{R}^n$ with radius 1 is called a **unit ball** and its boundary, denoted by $\partial B(0, 1)$, is the unit sphere. The **complement** of an open ball $B(x, r)$ is denoted by $B^c(x, r)$ and is defined as

$$B^c(x, r) := \{y \in \mathbb{R}^n : |x - y| \geq r\}.$$

Definition 2.1.3. Let $K \subset \mathbb{R}^n$ be a non-empty, non-convex close set. The **medial axis** M_K of set K is defined as a set of points in $\mathbb{R}^n$ such that every point $x \in M_K$ is equidistant from at least two points in K. In other words, for every point $x \in M_K$ there exists at least two points $y_1, y_2 \in K$, $y_1 \neq y_2$, such that $\operatorname{dist}(x, K) = |x - y_1| = |x - y_2|$. Mathematically,

$$M_K := \left\{x \in \mathbb{R}^n : \ \exists \ y_1, y_2 \in K, y_1 \neq y_2, \ \operatorname{dist}(x, K) = |x - y_1| = |x - y_2|\right\}$$

In the following Figure 2.2 we plot a triangle of points $\{k_1, k_2, k_3\}$ where the medial axis is the set of the perpendicular bisectors of points k_1, k_2, points k_1, k_3 and points k_2, k_3. The medial axis of the cricle is singleton set $\{0\}$ and the medial axis of the square is the set of every two points.

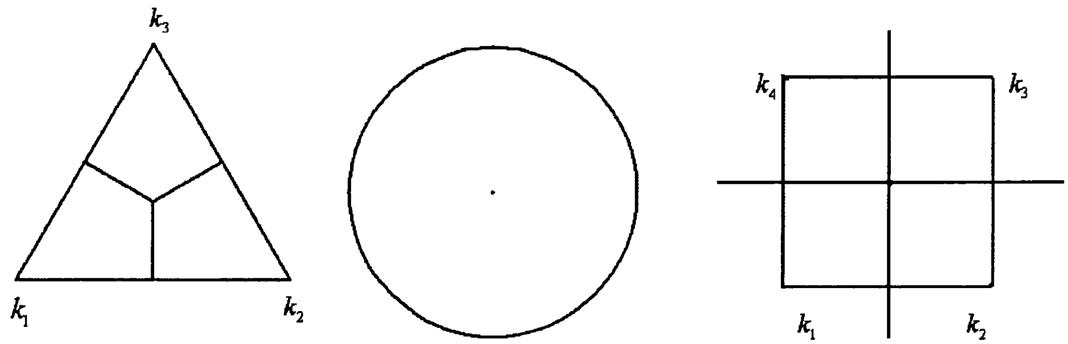

Figure 2.1: Firgure from left, the Medial axis of a Triangle of points $\{k_1, k_2, k_3\}$, Circle centred at 0 and Square of points $\{k_1, k_2, k_3, k_4\}$ in 2D.

Definition 2.1.4. We define $K(x) \subseteq K \subset \mathbb{R}^n$ a non-empty, non-convex and close set such that for every point $x \in \mathbb{R}^n$, set $K(x)$ is the set of points in K with minimum distance to x, that is,

$$K(x) := \left\{ y \in K : \ \mathrm{dist}(x, K) = \min_{y \in K} |x - y| \ \right\}$$

All the points of $K(x)$ are contained in the ball $\bar{B}(x, r(x))$ as boundary points, that is,

$$K(x) = \bar{B}\big(x, r(x)\big) \cap K,$$

where we define $r(x) := \mathrm{dist}(x, K(x)) = \min_{y \in K(x)} |x - y|$ for all $x \in \mathbb{R}^n$. If we suppose $x = 0$, then $r(0) = |y|$ for all $y \in K(x)$.

2.2 Theory of Convex Analysis

The space we are working in is the Euclidean space $\mathbb{R}^n$. Recall the following concepts of convexity from [17] and [18].

Definition 2.2.1. The set $K \subset \mathbb{R}^n$ is said to be **convex** if $\alpha x_1 + (1 - \alpha)x_2$ is in K whenever x_1, x_2 are in K, and $\alpha \in (0, 1)$ (or equivalently $\alpha \in [0, 1]$). Geometrically, this means that the line-segment

$$[x_1, x_2] := \{\alpha x_1 + (1 - \alpha)x_2 : 0 \leq \alpha \leq 1\}$$

is entirely contained in K whenever its end points x_1 and x_2 are in K.

Definition 2.2.2. An **affine combination** of a set of points $K = \{x_1, x_2, ..., x_n\} \subset \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$, where $\sum_{i=1}^{n} \alpha_i = 1$ and each α_i is a real number. If each α_i is nonnegative, the point x is a **convex combination** of K. The affine combination of K is called the **affine hull** of K, denoted by $aff\ K$ and the convex combination of K is called the **convex hull** of K, denoted by $C(K)$.

For instance, if we consider set K of three non-collinear points in $\mathbb{R}^2$, then the entire plane ($\mathbb{R}^2$) is the affine hull of K, and the triangle made by the vertices of these three points is the convex hull of K. The dimension of a convex set K is the dimension of the affine hull (which is a plane in this example).

Definition 2.2.3. The **relative interior** $ri\ K$ of a convex set $K \subset \mathbb{R}^n$ is the interior of K for the topology relative to the affine hull of K: $x \in ri\ K$ if and only if

$$x \in \text{aff } K \quad \text{and} \quad \exists \delta > 0 \ \text{ such that } \ (\text{aff } K) \cap B(x, \delta) \subset K.$$

Definition 2.2.4. A function $f(x) := ax + b$ for $x \in \mathbb{R}^n$, with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is called an **affine function** on $\mathbb{R}^n$.

Definition 2.2.5. Let the set K be a non-empty convex set in $\mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be **convex** on K when, for all $(x_1, x_2) \in K \times K$ and all $\alpha \in (0, 1)$, there holds

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2). \tag{2.2.1}$$

The function f is said to be strictly convex on K when 2.2.1 holds as a strict inequality if $x_1 \neq x_2$.

Definition 2.2.6. Convex envelope was given by the solution of a fully nonlinear, degenerate elliptic partial differential equationby A. M. Oberman [14]. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ bounded below, the **convex envelope** of f is defined as the supremum of all convex functions which are majorized by f. It is denoted by $C\big(f(x)\big)$ and mathematically,

$$C\big(f(x)\big) = \sup \big\{ g(x) : \ g \text{ convex}, \ g(y) \leq f(y), \ \forall y \in \mathbb{R}^n \big\}.$$

An equivalent definition from [4] is that, $C\big(f(x)\big)$ of function f can also be defined as the supremum of all affine functions ($l(x) = ax + b$) which are majorized by f,

$$C\big(f(x)\big) = \sup\big\{l(x) : \ l : \mathbb{R}^n \to \mathbb{R} \text{ affine}, \ l(y) \leq f(y), \ \forall y \in \mathbb{R}^n\big\}.$$

It can also be defined from [17, Corollary 17.1.5], let $f : \mathbb{R}^n \to (-\infty, +\infty]$, then

$$C\big(f(x)\big) = \inf\big\{\sum_{i=1}^{n+1} \lambda_i f(x_i) : \lambda_i \geq 0, x_i \in \mathbb{R}^n, \ i = 1, ..., n+1, \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i x_i = x\big\}.$$

Theorem 2.2.7. *[20] If $f : \mathbb{R}^n \to \mathbb{R}$ is of super-linear growth in the sense that*

$$\frac{f(x)}{|x|} \to +\infty \quad \textit{as } |x| \to \infty,$$

then there are $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, $x_i \in \mathbb{R}^n$ and $\sum_{i=1}^{n+1} \lambda_i x_i = x$ such that

$$C\big(f(x)\big) = \sum_{i=1}^{n+1} \lambda_i f(x_i).$$

Definition 2.2.8. Let $K \subset \mathbb{R}^n$. A function $f : K \to \mathbb{R}^n$ is **Lipschitz continuous** on K if for all points $x, y \in K$,

$$|f(x) - f(y)| \leq L|x - y|$$

where L is Lipschitz constant.

Definition 2.2.9. A function that has derivatives of all orders is called a smooth function whereas a function f is said to be $C^{1,1}$ **smooth** if f has Lipschitz continuous derivatives of order 1. In other words, a function f is of the class $C^{1,1}$ if the first derivative of f exists and is Lipschitz continuous.

Definition 2.2.10. [20, Definition 1.1]Suppose that function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfies

$$f(x) \geq -C_f|x|^2 - C_1, \qquad x \in \mathbb{R}^n, \tag{2.2.2}$$

for some constants $C_f > 0, C_1 > 0$, then the **quadratic lower compensated convex transform** (in short: **lower transform**) for f is defined by

$$C^l_\lambda(f(x)) = C\big(f(x) + \lambda|x|^2\big) - \lambda|x|^2, \ \ x \in \mathbb{R}^n, \ \ \text{for } \lambda > C_f,$$

where $C[f(x)+\lambda|x|^2]$ is the value of the convex envelope of the function $y \to f(y)+\lambda|y|^2$ at $x \in \mathbb{R}^n$. The lower transform has a translation invariance property, that means,

$$C_\lambda^l(f(x)) = C\big(f(x)+\lambda|x-x_0|^2\big) - \lambda|x-x_0|^2$$

implies that

$$C_\lambda^l(f(x)) = C\big(f(x)+\lambda|y-x|^2\big)\Big|_{y=x_0}.$$

Definition 2.2.11. [20] Suppose $f:\mathbb{R}^n \to \mathbb{R}$ satisfies $f(x) \le C_f|x|^2 - C_1$, $x \in \mathbb{R}^n$, for some constants $C_f > 0, C_1 > 0$, then the quadratic upper compensated convex transform (in short upper transform) for f is defined by

$$C_\lambda^u(f(x)) = -C_\lambda^l[-f(x)], \quad x \in \mathbb{R}^n, \quad \text{for } \lambda > C_f,$$

or

$$C_\lambda^u(f(x)) = -C[-f(x)+\lambda|x|^2]+\lambda|x|^2, \quad x \in \mathbb{R}^n, \quad \text{for } \lambda > C_f,$$

where $C[-f(x)+\lambda|x|^2]$ is the value of the convex envelope of the function $y \to -f(y)+\lambda|y|^2$ at $x \in \mathbb{R}^n$.

Definition 2.2.12. The **gradient** of the lower transform $h(x) := C_\lambda^l \mathrm{dist}^2(x,K)$ of squared distance function to set $K \subset \mathbb{R}^n$ is

$$Dh(x) = \Big(\frac{\partial h}{\partial x_1}(x), \frac{\partial h}{\partial x_2}(x), \dots \frac{\partial h}{\partial x_n}(x)\Big)$$

The gradient of the lower transform of the squared distance function to set $K \subset \mathbb{R}^2$ is

$$Dh(x_1,x_2) = \Big(\frac{\partial h}{\partial x_1}(x_1,x_2), \frac{\partial h}{\partial x_2}(x_1,x_2)\Big) = \big(h_{x_1}(x_1,x_2), h_{x_2}(x_1,x_2)\big)$$

where $x := (x_1,x_2) \in \mathbb{R}^2$. Furthermore, the vector g is called a **subgradient** of a function $f:\mathbb{R}^n \to \mathbb{R}$ at x if

$$f(y) \ge f(x) + g\cdot(y-x)$$

for all $y \in \mathbb{R}^n$.

Definition 2.2.13. The **directional derivative** $D_{\hat{e}}h$ of the lower transform of the squared distance functions in the direction of $\hat{e}$ is $\nabla_e f(x_0,y_0,z_0)$ is the rate at which the function $f(x, y, z)$ changes at a point (x_0, y_0, z_0) in the direction e. It is a vector form of the usual derivative, and can be defined as

$$\begin{aligned} D_{\hat{e}} C_\lambda^l \mathrm{dist}^2(x, K) &= \nabla f \cdot \hat{e} \\ &= \lim_{t \to 0} \frac{f(x + t\hat{e}) - f(x)}{t} \end{aligned}$$

Definition 2.2.14. [14] A **supporting hyperplane** to a function $f : \mathbb{R}^n \to \mathbb{R}$ at the point x is a plane $P(y) = f(x) + g \cdot (y - x)$, which touches the graph of function f from below, i.e., which satisfies

$$f(y) \geq f(x) + g \cdot (y - x)$$

for all $y \in \mathbb{R}^n$, where the vector g is a subgradient of f at x.

We provide preliminaries from [20] concerning the lower transform $C_\lambda^l(f)$, and in particular, the lower transform $C_\lambda^l \mathrm{dist}^2(x, K)$ for the squared distance functions to set $K \subset \mathbb{R}^n$, that will be helpful later. One of the properties of the lower transform $C_\lambda^l(f)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 2.2.2 is called the 'tight approximation' of the original function f. We provide the following theorem from [20, Theorem 2.3] concerning the effects of continuity, smoothness and local geometry of the original function f on the convergence of the lower transform $C_\lambda^l(f)$ to the original function f as $\lambda \to +\infty$.

Theorem 2.2.15. *[20, Theorem 2.3] Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies 2.2.2.*

(i) If $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous, then for every $x \in \mathbb{R}^n$,

$$\lim_{\lambda \to +\infty} C_\lambda^l\big(f(x)\big) = f(x) \tag{2.2.3}$$

(ii) Assume that $x_0 \in \mathbb{R}^n$ is a local minimum point of f, then there is some $\lambda_{x_0} > 0$ such that $C_\lambda^l\big(f(x_0)\big) = f(x_0)$ whenever $\lambda \geq \lambda_{x_0}$.

(iii) If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous in $\mathbb{R}^n$, then 2.2.3 holds uniformly on any compact subset of $\mathbb{R}^n$.

(iv) If for some $x_0 \in \mathbb{R}^n$, $f \in C^{1,1}\big(\bar{B}(x_0, \delta)\big)$ for some $\delta > 0$, then

$$C_\lambda^l\big(f(x_0)\big) = f(x_0), \quad \textit{whenever } \lambda > \max\Big\{C(x_0), 4C_f' + \frac{1}{\delta^2}\big(2C_f' + 1 + |Df(x_0)|\big)\Big\},$$

where $C(x_0) > 0$ such that $|Df(x) - Df(x_0)| \le C(x_0)|x - x_0|$ whenever $|x - x_0| \le \delta$, and $f(y) \ge -C_f'(|y|^2 + 1)$.

(v) If $f \in C^{1,1}(\mathbb{R})$ and $L > 0$ be such that $|Df(y) - Df(x)| \le L|y - x|$ for all $x, y \in \mathbb{R}^n$, then $C_\lambda^l(f(x)) = f(x)$ for all $x \in \mathbb{R}^n$ whenever $\lambda \ge L$.

Remark: This shows that the analytic property that f is $C^{1,1}$ locally guarantees that $f(x)$ can be obtained in finite time (that is, for a finite value of λ) by the quadratic lower transforms, and thus we may call this property the 'tight approximation' of the original function f. This property of compensated convex transforms is not shared by any other well-known smooth approximations such as mollifier smoothing.

Theorem 2.2.16. *[17, Caratheodory's Theorem 17.1] Let S be any set of points and directions in $\mathbb{R}^n$, and $C(S)$. Then $x \in C(S)$ if and only if x can be expressed as a convex combination of $n + 1$ of the points and directions in S (not necessarily distinct). In fact $C(S)$ is the union of all the generalized d-dimensional simplices whose vertices belong to S, where $d = \dim C(S)$.*

An equivalent statement. Let $K \subset \mathbb{R}^n$, then each $x \in C(K)$ may be written as a convex combination of (say) m affine independent points in K (not necessarily distinct). In particular, $m \le n+1$, that is, $x \in C(K)$, there are at most $n+1$ points, $x_1, x_2, \ldots, x_{(n+1)} \in K$, such that $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$, with $\lambda_i \ge 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$.

Theorem 2.2.17. *[20, Theorem 3.1] Suppose $K \subset \mathbb{R}^n$ is non-empty and compact. Then $C_\lambda^l dist^2(x, K) \in C^{1,1}(\mathbb{R}^n)$ and the Lipschitz constant for the gradient $DC_\lambda^l dist^2(x, K)$ is bounded above by $8 + 10\lambda$. Futhermore, $C_\lambda^l dist^2(x, K) = 0$ if and only if $x \in K$ and $\lambda > 0$.*

Lemma 2.2.18. *[20, Lemma 3.2] Let $K = B^c(x, r(x)) = \{y : \ y \in \mathbb{R}^2, |y - x| \ge r\}$ be the complement of the open ball $B(x, r(x))$ with $r > 0$, then*

$$C_\lambda^l dist^2(x, K) = \begin{cases} \frac{\lambda}{1+\lambda} r^2 - \lambda |x|^2, & |x| \le \frac{r}{1+\lambda} \\ dist^2(x, K), & |x| \ge \frac{r}{1+\lambda} \end{cases}$$

Lemma 2.2.19. *[20, Lemma 3.3] Let $n \ge 2$ and let $\{e_1, \ldots, e_n\}$ be the standard Euclidean basis of $\mathbb{R}^n$, where e_i is the vector with its ith component 1 and others zero. Let $K =$*

$\{-\alpha e_1, \alpha e_1\}$, where $\alpha > 0$. We write $y = e_2 y_2 + ... + e_n y_n \in \mathbb{R}^{n-1}$ and $xe_1 + y := (x, y) = (x, y_2, ..., y_n) \in \mathbb{R}^n$ with $x, y_i \in \mathbb{R}$, $2 \leq i \leq n$. Then for every $\lambda > 0$, we have

$$C_\lambda^l(dist^2((x,y),K)) = \begin{cases} \frac{\lambda}{1+\lambda}\alpha^2 - \lambda x^2 + |y|^2, & |x| \leq \frac{\alpha}{1+\lambda} \\ dist^2((x,y),K), & |x| \geq \frac{\alpha}{1+\lambda} \end{cases}$$

Lemma 2.2.20. *[2, Corollary 2.5] [20, Lemma 4.2] Suppose $h : B(\hat{x}_0, r) \to \mathbb{R}$ is convex and $g : B(\hat{x}_0, r) \to \mathbb{R}$ is upper semi-differentiable at $\hat{x}_0$ such that $h \leq g$ on $B(\hat{x}_0, r)$ and $h(\hat{x}_0) = g(\hat{x}_0)$. Then h and g are both differentiable at $\hat{x}_0$ and $Dh(\hat{x}_0) = Dg(\hat{x}_0)$.*

Theorem 2.2.21. *[16, Theorem 2.5 (Arzela-Ascoli)] Let X be a compact subset of $\mathbb{R}^{m_1}$, and let $\{f_n\}$ be a sequence of continuous functions from X to $\mathbb{R}^{m_2}$. If $\{f_n\}$ is uniformly bounded, that is, there exists an M such that*

$$\|f_n\|_\infty \leq M \quad \text{for all } n,$$

and equi-continuous, that is, for every $\epsilon > 0$ there exists a $\delta > 0$, independent of n, such that

$$|x - y| \leq \delta \ \text{ implies that } \quad |f_n(x) - f_n(y)| \leq \epsilon,$$

then $\{f_n\}$ has a subsequence that converges uniformly on X.

2.3 Morse Indices

In this section we provide definitions of geometric critical points and Morse indices of critical points in $\mathbb{R}^3$. The definition of Morse indices of critical points are defined in [13], using the Morse Lemma [13, Lemma 2.2], which is also included in this section as Lemma 2.3.2, by which it can be seen that the behaviour of a smooth function at the critical point can be described by the index of the function. We further provide definitions of non-degenerate critical points and degenerate critical points.

Definition 2.3.1. Let $K \subset \mathbb{R}^n$ be a non-empty, non-convex closed set. Then the critical point x_0 is called a **geometric critical point** of the squared distance function if x_0 belongs to the convex hull of K (i.e. $x_0 \in C(K)$).

Lemma 2.3.2. *[13, Morse Lemma 2.2] Let p be a non-degenerate critical point for $f \in C^2$. Then there is a local coordinate system $(x_1, ..., x_n)$ in a neighbourhood U of p with $x_i(p) = 0$ for all i and such that the identity*

$$f = f(p) - (x_1)^2 - ... - (x_\mu)^2 + (x_{\mu+1})^2 + ... + (x_n)^2 \tag{2.3.4}$$

holds throughout U. The Morse index of f at p is defined to be μ.

Definition 2.3.3. [6]The **non-degenerate critical points** are characterised by the Morse Lemma. This characterisation is based on the number of minus signs in this expression (2.3.4), called the index μ of f at p. For each non-degenerate critical point p, there is a local coordinate system with the origin at p so that

$$f(y) = f(p = 0) \pm (x_1)^2 \pm (x_2)^2 \pm (x_3)^2$$

for all $y = (x_1, x_2, x_3)$ in a neighbourhood of p. The number of minus signs in this expression is the index of p. There are three categories :

- The non-degenerate critical points of index 0 are local minima.
- The non-degenerate critical points of index 3 are the local maxima.
- The non-degenerate critical points of index 1 or 2 are saddle points.

Definition 2.3.4. [6]A **manifold** is a topological space that is locally Euclidean, that is, around every point, there is a neighbourhood that is topologically the same as the open unit ball in $\mathbb{R}^n$. A **stable manifold**, denoted by S of the critical point x_0 of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$S(x_0) = \{x_0\} \cup \{y \in \mathbb{R}^3 : \text{ the solution } \phi \text{ of } \dot{\phi}(t) = -Df\big(\phi(t)\big),\ \phi(0) = y, \text{ converges to } x_0 \text{ as } t \to \infty\}.$$

Throughout our thesis we will not used the negative gradient but instead use $\dot{\phi} = Df(\phi)$.

Definition 2.3.5. For every point $x \in \mathbb{R}^3$, let $q(x) \in C\big(K(x)\big)$ be the closest point to x, then point $q(x)$ is called a driver of x.

Lemma 2.3.6. *[6, Flow Lemma 10.1] For any regular point $x \in \mathbb{R}^3$ let $q(x)$ be a driver of x. The steepest ascent of the distance function at x is in the direction of $x - q(x)$.*

Definition 2.3.7 (Polytope)**.** [6] A k-polytope is the convex hull of a set of points which has at least $(k+1)$ affinely independent points.

2.4 Complexes

We need geometric structures for surface reonstruction, such as simplicial complexes and Delaunay complexes of finite sets in $\mathbb{R}^3$. We will consult [6] for basic preliminaries for surface reconstructions and we denote convex hull of set K by $C(K)$.

Definition 2.4.1. A **k-simplex** σ is the convex hull of exactly $(k+1)$ affinely independent points. There are at most four types of simplices in $\mathbb{R}^3$: vertices or 0-simplices, edges or 1-simplices, triangles or 2-simplices and tetrahedra or 3-simplices. A **circumscribing ball** of a simplex σ is a ball whose boundary contains the vertices of the simplex. A ball is K-empty if its interior does not contain any point from set K.

Definition 2.4.2. A simplex $\sigma = C(T)$ for a nonempty subset T of finite set K is called a **face** of $\sigma' = C(K)$ and σ' is a **coface** of σ.

Definition 2.4.3. A collection κ of simplices is called a **simplicial complex** if:

(i) $\sigma \in \kappa$ if σ is a face of any simplex $\sigma' \in \kappa$,

(ii) For any two simplices $\sigma, \sigma' \in \kappa$, their intersection is a face of both unless it is empty.

A **cell complex** is a collection of polytopes and their face where any two intersecting polytopes meet in a face which is also in the collection. The **k-complex** is a cell complex if the largest dimension of any polytope in the collection is k. It is exactly the same as simplicial complex where only simplices are replace by polytopes.

Definition 2.4.4. A **Voronoi diagram** Vor K of a finite point set K of points partitions the space into cells, called Voronoi cells V_p of the given finite set K. The **voronoi cell**

V_p is the set of all points in the plane that have no other points in K closer to it than p. It can be written as for each point $p \in K$ we define V_p as

$$V_p = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, K) = |x - p|\}.$$

A k-dimensional **voronoi face** for $k \leq 2$ is the set of all points that are equidistant from $(3 - k)$ points in K. In other words, a k-dimensional voronoi face is the intersection of $3 - k$ voronoi cells. Thus, a Voronoi diagram Vor K is the cell complex formed by the voronoi faces. A 0-dimensional voronoi face which is equidistant from 3 points in K is called **Voronoi vertex** and a 1-dimensional voronoi face which contains points equidistant from 2 points in K is called a **Voronoi edge**.

Definition 2.4.5. The **Delaunay triangulation** Del K of finite point set K is the dual of the Voronoi daigram Vor K. It is defined as a simplicial complex such that

$$\text{Del } K = \Big\{\sigma = C(T) \mid \bigcap_{p \in T \subseteq K} V_p \neq \emptyset\Big\}.$$

A **Delaunay k-simplex** in Del K is formed by $(k+1)$ points in K if their Voronoi cells Vor K have nonempty intersection. The emptiness property of Delaunay triangulation which is applied to triangles and edges:

(i) Triangle emptiness property: A triangle is in the Delaunay triangulation if and only if its circumscribing ball is empty.

(ii) Edge emptiness property: An edge is in the Delaunay triangulation if and only if the edge has an empty circumscribing ball.

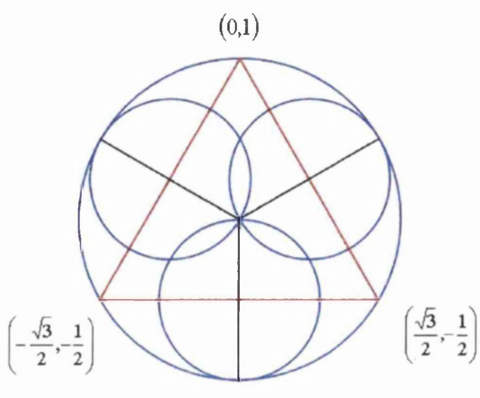

Figure 2.2: Delaunay trianlge and edges of $K = \{(\frac{\sqrt{3}}{2}, -\frac{1}{2}), (0,1), (-\frac{\sqrt{3}}{2}, -\frac{1}{2})\}$ in 2D.

Chapter 3

Critical points and Morse indices for smoothed squared distance functions

In this chapter we establish a key geometric critical point theorem of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ (i.e., the tight approximation) of the squared-distance function to a compact set K in $\mathbb{R}^n$. The result given by Kewei Zhang [20] have only shown that, a medial axis point in a bounded domain $\Omega \in \mathbb{R}^2$ under certain assumptions is critical point of the smoothed squared distance function at the boundary of the bounded domain. On the other hand, the main feature of our new key geometric critical point theorem (Theorem 3.1.1) is that the non-smooth geomteric critical points of the squared distance functions and critical points of the smoothed squared distance function are the same. Other types of smoothing might not usually have these types of properties which is a benefit of our smooth approximation. For instance, it is not clear whether the convolutions molifier smoothing has the property that the non-smooth critical points of the squared distance function are the critical points for smooth function [9]. Then we plot the squared distance function to different finite sets and their smooth squared distance function (the lower transforms) to illustrate that smooth and non-smooth critical points are the same. In the next section we illustrate critical points with Morse indices with the help of some prototype examples for the tight approximation of finite sets in two and three dimensions.

Typical situations for the critical points with Morse indices in two and three dimensions are explained and then we classify the critical points with Morse indices for finite sets in $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.1 Geometrical Critical Point Theorem

Theorem 3.1.1. *Let K be a non-empty, non-convex closed subset of $\mathbb{R}^n$. A point $x \in M_K \setminus K$ is a critical point of $C^l_\lambda dist^2(x, K)$, that is, $DC^l_\lambda dist^2(x, K) = 0$, if and only if x is a geometric critical point of K.*

The definition of medial axis M_K is given by Definition 2.1.3 in Chapter 2. In this Theorem 3.1.1, we prove that the non-smooth geometric critical points of the squared distance functions are the same as the usual critical points of the smoothed squared distance functions. Before we prove Theorem 3.1.1, let us first apply the lower transform to the squared distance $\mathrm{dist}^2\big(x, K(x_0)\big)$ to non-emtpy, non-convex closed set $K(x_0) \subseteq K \subset \mathbb{R}^n$ when $x_0 \in M_{K(x_0)}$. The following derived explicit $C^{1,1}$-approximation formula for the squared distance functions will be used later in this chapter to calculate the explicit formulae for the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ for the squared distance function to finite sets in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

Lemma 3.1.2. *Let K be a non-empty, non-convex closed subset of $\mathbb{R}^n$ that contains at least two elements. Let $x_0 \in M_K$ and define $r(x_0) := dist(x_0, K(x_0))$. Then for all $y \in \mathbb{R}^n$*

$$C^l_\lambda dist^2\big(y, K(x_0)\big) = \frac{\lambda}{(1+\lambda)} r^2 + (1+\lambda) dist^2\Big(y, \frac{C(K(x_0))}{1+\lambda}\Big) - \lambda |y - x_0|^2 \qquad (3.1.1)$$

Proof. Suppose for simplicity, that $x_0 = 0$. We can see that $\mathrm{dist}^2\big(y, K(0)\big) = \min_{y' \in K(0)} |y - y'|^2$. Then for all $y' \in K(0)$, we have

$$\begin{aligned} \mathrm{dist}^2\big(y, K(0)\big) + \lambda |y|^2 &= \min_{y' \in K(0)} \Big\{ |y - y'|^2 + \lambda |y|^2 \Big\} \\ &= \frac{\lambda}{(1+\lambda)} |y'|^2 + (1+\lambda) \min_{y' \in K(0)} \Big\{ \Big| y - \frac{y'}{1+\lambda} \Big|^2 \Big\} \end{aligned}$$

The convex envelope of $\mathrm{dist}^2\big(y, K(0)\big) + \lambda |y|^2$ is then

$$C\big(\mathrm{dist}^2\big(y, K(0)\big) + \lambda |y|^2\big) = \frac{\lambda}{(1+\lambda)} r^2(0) + (1+\lambda) \mathrm{dist}^2\Big(y, \frac{C(K(0))}{1+\lambda}\Big).$$

Therefore,

$$C^l_\lambda \text{dist}^2\big(y, K(0)\big) = \frac{\lambda}{(1+\lambda)} r^2(0) + (1+\lambda)\text{dist}^2\Big(y, \frac{C(K(0))}{1+\lambda}\Big) - \lambda|y|^2,$$

and thus the lower transform of the squared distance function to $K(x_0)$ when $x_0 \in M_K$ can be written as

$$C^l_\lambda \text{dist}^2\big(y, K(x_0)\big) = \frac{\lambda}{(1+\lambda)} r^2(x_0) + (1+\lambda)\text{dist}^2\Big(y, \frac{C(K(x_0))}{1+\lambda}\Big) - \lambda|y - x_0|^2$$

for all $y \in \mathbb{R}^n$. □

Proof of Theorem 3.1.1. Suppose that a point x is a geometric critical point of K, that is, x belongs to the convex hull of $K(x)$, i.e., $x \in C\big(K(x)\big)$. We have to show that x is a critical point of $C^l_\lambda \text{dist}^2(x, K)$, that is, $DC^l_\lambda(\text{dist}^2(x, K)) = 0$.
Without loss of generality let us assume that $x = 0$. From Theorem 2.2.17, we know that $C^l_\lambda \text{dist}^2(\cdot, K) \in C^{1,1}(\mathbb{R}^n)$. Furthermore, it is clear that $K(0) \subset K \subset B^c\big(0, r(0)\big)$. Then clearly, for all $y \in \mathbb{R}^n$, we have

$$\text{dist}^2\big(y, B^c(0, r(0))\big) \leq \text{dist}^2(y, K) \leq \text{dist}^2\big(y, K(0)\big).$$

The lower transform preserves inequalities [20, Theorem 2.1 (ii)], so applying the lower transform gives

$$C^l_\lambda \text{dist}^2\big(y, B^c(0, r(0))\big) \leq C^l_\lambda \text{dist}^2(y, K) \leq C^l_\lambda \text{dist}^2\big(y, K(0)\big), \tag{3.1.2}$$

for all $y \in \mathbb{R}^n$. From equation (3.1.1) for $y = 0$, we have

$$C^l_\lambda \text{dist}^2\big(0, K(0)\big) = \frac{\lambda}{(1+\lambda)} r^2. \tag{3.1.3}$$

We also know from Lemma 2.2.18 for $y = 0$ that

$$C^l_\lambda \text{dist}^2\big(0, B^c(0, r(0))\big) = \frac{\lambda}{(1+\lambda)} r^2. \tag{3.1.4}$$

Therefore, (3.1.2) for $y = 0$ implies that

$$C^l_\lambda \text{dist}^2\big(0, B^c(0, r(0))\big) = C^l_\lambda \text{dist}^2(0, K) = C^l_\lambda \text{dist}^2\big(0, K(0)\big). \tag{3.1.5}$$

Let $e \in \mathbb{R}^n$ and $|e| = 1$ be a unit direction vector. Then from Lemma 2.2.18 we have

$$C_\lambda^l \text{dist}^2\big(te, B^c(0, r(0))\big) = \frac{\lambda}{(1+\lambda)} r^2(0) - \lambda |te|^2. \tag{3.1.6}$$

Therefore, for a small $t > 0$, we have

$$\frac{C_\lambda^l \text{dist}^2\big(et, B^c(0, r(0))\big) - C_\lambda^l \text{dist}^2\big(0, B^c(0, r(0))\big)}{t} = \frac{-\lambda |te|^2}{t} \tag{3.1.7}$$

From inequality (3.1.2), equation (3.1.5) and for small $t > 0$, we know that

$$\frac{C_\lambda^l \text{dist}^2\big(et, B^c(0, r(0))\big) - C_\lambda^l \text{dist}^2\big(0, B^c(0, r(0))\big)}{t} \le \frac{C_\lambda^l \text{dist}^2(et, K) - C_\lambda^l \text{dist}^2(0, K)}{t}.$$

This, together with (3.1.7), implies that

$$\frac{-\lambda |te|^2}{t} \le \frac{C_\lambda^l \text{dist}^2(te, K) - C_\lambda^l \text{dist}^2(0, K)}{t}.$$

If we take the limit $t \to 0_+$, then

$$0 \le DC_\lambda^l \text{dist}^2(0, K) \cdot e$$

Suppose that $e_1 = -e$, then we have

$$0 \ge DC_\lambda^l \text{dist}^2(0, K) \cdot e$$

Therefore, for all $e \in \mathbb{R}^n$ and $|e| = 1$, we have

$$DC_\lambda^l \text{dist}^2(0, K) \cdot e = 0,$$

so that

$$DC_\lambda^l \text{dist}^2(0, K) = 0,$$

that is, 0 is a critical point of the lower transform of squared-distance function, $C_\lambda^l \text{dist}^2(\cdot, K)$, at $y = 0$.

Conversely, if x is a critical point of $C_\lambda^l \text{dist}^2(x, K)$, then we prove that $x \in C\big(K(x)\big)$. Let us assume without loss of generality that $x = 0$. From the definition of lower transform 2.2.10 at $x = 0$, we have

$$C_\lambda^l \text{dist}^2(x, K) = C\big\{\text{dist}^2(x, K) + \lambda |x|^2\big\}\Big|_{x=0}. \tag{3.1.8}$$

We have that $x = 0$ is a critical point of $C^l_\lambda \mathrm{dist}^2(x, K)$, that is,

$$DC^l_\lambda \mathrm{dist}^2(x, K) = 0.$$

Hence, from equation (3.1.8), this implies that

$$\begin{aligned} D\Big(C\{\mathrm{dist}^2(x, K) + \lambda|x|^2\}|_{x=0}\Big) &= 0 \\ C\{\mathrm{dist}^2(x, K) + \lambda|x|^2\}\big|_{x=0} &= c, \end{aligned} \tag{3.1.9}$$

for some $c \in \mathbb{R}^n$. Let $l : \mathbb{R}^n \to \mathbb{R}$ be the supporting affine function of $\mathrm{dist}^2(x, K) + \lambda|x|^2$, such that

$$l(x) \leq \mathrm{dist}^2(x, K) + \lambda|x|^2 \tag{3.1.10}$$

and

$$l(0) = C\{\mathrm{dist}^2(x, K) + \lambda|x|^2\}\big|_{x=0}.$$

Then there exist $\lambda_1, ..., \lambda_k > 0$, $\sum_{i=1}^k \lambda_i = 1$, $2 \leq k \leq n+1$, $x_1, ..., x_k \in \mathbb{R}^n$ and $\sum_{i=1}^k \lambda_i x_i = 0$, such that

$$l(x_i) = \mathrm{dist}^2(x_i, K) + \lambda|x_i|^2. \tag{3.1.11}$$

We can write $l(x)$ by

$$l(x) = a.x + c$$

where

$$l(0) = c = C\{\mathrm{dist}^2(x, K) + \lambda|x|^2\}\big|_{x=0}.$$

Since $\mathrm{dist}^2(x, K) + \lambda|x|^2$ is upper semi-differentiable and from inequality (3.1.10), equation (3.1.11) and Lemma 2.2.20 we have

$$Dl(x)\big|_{x=x_i} = D\{\mathrm{dist}^2(x, K) + \lambda|x|^2\}\big|_{x=x_i} \tag{3.1.12}$$

and $Dl(x) = a$, therefore,

$$Dl(0) = DC\{\mathrm{dist}^2(x, K) + \lambda|x|^2\}\big|_{x=0} = 0.$$

Thus $a = 0$ and (3.1.10) and (3.1.11) then yield that $c \leq \mathrm{dist}^2(x, K) + \lambda|x|^2$, for all $x \in \mathbb{R}^n$ and

$$\mathrm{dist}^2(x_i, K) + \lambda|x_i|^2 = c.$$

Moreover, choose $y_i \in K$ such that $|x_i - y_i|^2 = \text{dist}^2(x_i, K)$ and note that, since

$$l(x) \leq \text{dist}^2(x, K) + \lambda|x|^2 \leq \text{dist}^2(x, y_i) + \lambda|x|^2,$$

and

$$l(x_i) = \text{dist}^2(x_i, K) + \lambda|x_i|^2 = |x_i - y_i|^2 + \lambda|x_i|^2,$$

it follows from Lemma 2.2.20 that

$$D\{|x - y_i|^2 + \lambda|x|^2\}\big|_{x=x_i} = Dl(x_i),$$

and hence $2(x - y_i) + \lambda x_i = 0$, so that $y_i = (1 + \lambda)x_i$ and so

$$\begin{aligned} |x_i - y_i| &= \lambda|x_i| \\ |x_i| &= \frac{|y_i|}{(1+\lambda)} \end{aligned} \tag{3.1.13}$$

Therefore, using equation (3.1.12) and (3.1.13) we have

$$\begin{aligned} \text{dist}^2(x_i, K) + \lambda|x_i|^2 &= l(x_i) \\ |x_i - y_i|^2 + \lambda|x_i|^2 &= c \\ \left|\frac{y_i}{(1+\lambda)} - y_i\right|^2 + \frac{\lambda|y_i|^2}{(1+\lambda)^2} &= c \\ \frac{\lambda|y_i|^2}{(1+\lambda)} &= c \end{aligned}$$

We define $r^2 := \frac{1+\lambda}{\lambda}c$, then $y_i \in \partial B(0, r)$, for all $i = 1, 2, ..., k$. Now in order to show that $x = 0 \in C[K(0)]$, we have to show that inside the ball $B(0, r)$, there are no other points of K. We show this using a contradiction argument.

Let us assume that there is a point $y^* \in K$, such that point y^* is in $B(0, r)$, that is, $|y^*| < |y_i|$ for all $i = 1, 2, ...k$. Define $x^* = \frac{y^*}{1+\lambda}$, then

$$\begin{aligned} \text{dist}^2(x^*, K) + \lambda|x^*|^2 &\leq |x^* - y^*|^2 + \lambda|x^*|^2 \\ &= \frac{\lambda}{1+\lambda}|y^*|^2 < \frac{\lambda}{1+\lambda}|y_i|^2 = c \end{aligned}$$

This contradicts to

$$c \leq \text{dist}^2(x, K) + \lambda|x|^2$$

for all $x \in \mathbb{R}^n$. We have proved that $K \cap B(0, r(0)) = \emptyset$, that is, there are no points of K other than the points in $K(0) = K \cap \partial B(0, r(0))$.

As we have shown that there is no point of K in $B(0, r(0))$, so

$$K \cap \partial B(0, r(0)) = \bar{B}(0, r(0)) \cap K.$$

The intersection of K with $\bar{B}(0, r(0))$ consists only of the points of $K(0)$. Hence we have

$$K \cap \partial B(0, r(0)) = K(0).$$

Also, since $\sum_{i=1}^{k} \lambda_i x_i = 0$, and $y_i = (1+\lambda)x_i$, $y_i \in K(0)$, so that $\sum_{i=1}^{k} \lambda_i y_i = 0$. Therefore we have shown that $x = 0 \in C(K(0))$. □

3.2 Classification of Morse indices of Critical Points for finite sets in $\mathbb{R}^2$ and $\mathbb{R}^3$

In this section we classify the Morse indices of the critical points of the lower transform $C_\lambda^l dist^2(\cdot, K)$ of the squared distance function to a finite set $K \subset \mathbb{R}^n$, where $n = 2$ and $n = 3$. The motivation for exploring Morse indices is that in the context of surface reconstruction, the index gives the dimension of the stable manifold of the corresponding dynamical system, which is then used to reconstruct the surface. The idea of the reconstruction in 3D space is that the reconstructed surface is the collection of the stable manifolds of the index 2 critical points. Based on the geometric critical point Theorem 3.1.1 and locality properties of $C_\lambda^l dist^2(\cdot, K)$, we will classify critical points locally. Thus, for a critical point x_0, we only need to consider $K(x_0) \subset \mathbb{R}^n$ where $n = 2$ and $n = 3$. In this part we set $x_0 = 0$ and $K = K(0)$, as we only consider $\frac{1}{2}C_\lambda^l dist^2(y, K(0))$ near 0 in $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ and $\|x\| = (x_1^2 + \ldots, x_n^2)^{\frac{1}{2}}$ denote its distance from the origin. Then from Dey's Book [6], the definition of n-sphere $\mathbb{S}^n$ for $r > 0$ is

$$\mathbb{S}^n(0, r) = \{x \in \mathbb{R}^{n+1} : \|x\| = r\},$$

and in particular, we denote circle (1-sphere) by $\mathbb{S}^1$ and 2-sphere by $\mathbb{S}^2$ throughout this thesis. Non-degenerate critical points are characterized by the famous Morse Lemma for a C^2 function $h : \mathbb{R}^3 \to \mathbb{R}$.

Definition 3.2.1 (Non-degenerate Critical Points)**.** For each critical point x, there is a local coordinate system with the origin at x so that

$$h(y) = h(x = 0) \pm x_1^2 \pm x_2^2 \pm x_3^2$$

for all $y = (x_1, x_2, x_3)$ in a neighbourhood of x. The number of minus signs in this expression is the index of x.

We now define the indices of critical points in the context of the lower transform $C_\lambda^l \mathrm{dist}^2(x, K)$ of the squared distance function to a finite set K, which is $C^{1,1}$ or (non-smooth squared distance functions).

Definition 3.2.2. If x is a relative interior point of $C\big(K(x)\big)$ and $\dim C\big(K(x)\big) = k$, then x is called the **Morse index k geometric non-degenerate critical point** of the lower transform $C_\lambda^l \mathrm{dist}^2(x, K)$ of the squared distance function to a finite set K.

Definition 3.2.3. If x is a relative boundary point of the $C\big(K(x)\big)$, then x is called a **geometric degenerate critical point** of the lower transform $C_\lambda^l \mathrm{dist}^2(x, K)$ of the squared distance function to a finite set K.

Definition 3.2.4. (i) If x is a relative boundary point of $C\big(K(x)\big)$ with $K \subset \mathbb{R}^2$ and the Hessian of $\frac{1}{2} C_\lambda^l \mathrm{dist}^2\big(x, K\big)$ in the relative interior of $C\big(K(x)\big)$ has two negative eigenvalues and in the exterior the Hessian has one negative eigenvalue in the neighbourhood of x, then we call x a critical point with Morse index $(1, 2)$.

(ii) If x is a relative boundary point of $C\big(K(x)\big)$ with $K \subset \mathbb{R}^3$ and the Hessian of $\frac{1}{2} C_\lambda^l \mathrm{dist}^2\big(x, K\big)$ for x in the relative interior of $C\big(K(x)\big)$ has three negative eigenvalues and in the exterior the Hessian has both one and two negative eigenvalues in the neighbourhood of x, then we call x a critical point with Morse index $(1, 2, 3)$.

Definition 3.2.5. The **critical points** or equilibrium solutions of the first order gradient system

$$\dot{x} = D\frac{1}{2}C^l_\lambda dist^2(x, K) \tag{3.2.14}$$

are the points where the right hand side of (3.2.14) is zero. If the solutions converge towards a critical point as t increases, then the critical point is called **stable**. A small perturbation of its initial condition do not affect the longtime behaviour of these solutions (see examples of non-degenerate critical points of lower transform in Chapter 5). If the solutions diverge away from a critical point as t increases, then the critical point is called **unstable**. The behaviour of such solutions varies significantly even if the initial condition gets a small perturbation (see examples of the lower transform $C^l_\lambda dist^2(x, K)$ where the critical points are degenerate critical points in Chapter 5).

Remark 3.2.6. Note that for any $p = (p_x, p_y) \in K$ if we assume that $p_x < 0$ this means that if there are points of set K then these points can only be on the negative x-axis and vice versa and similarly for p_y and p_z. Further for any $p = (p_x, p_y, p_z) \in K$ if we assume that $p_x < 0$ and $p_y < 0$ this means that if there are points of set K then these points can only be on the negative xy plane and vice versa.

Theorem 3.2.7 (Classification of Critical Points for $\mathbb{R}^2$). *Suppose $K \subset \mathbb{S}^1\big((0,0), r\big)$ is finite and $(0,0) \in C(K)$.*

(I) If $\dim C(K) = 1$, then $\frac{1}{2}C^l_\lambda dist^2\big((x,y), K\big)$ is C^∞ near 0 and

$$\frac{1}{2}C^l_\lambda dist^2\big((x,y), K\big) = \frac{\lambda}{2(1+\lambda)}r^2 + \frac{1}{2}x^2 - \frac{\lambda}{2}y^2$$

near $(0,0)$ and the critical point $(0,0)$ is a standard (non-degenerate) critical point with Morse index 1.

(II) If $\dim C(K) = 2$ and $(0,0)$ is an interior point of $C(K)$. Then

$$\frac{1}{2}C^l_\lambda dist^2\big((x,y), K\big) = \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}x^2 - \frac{\lambda}{2}y^2$$

near $(0,0)$ and the critical point $(0,0)$ is a standard (non-degenerate) critical point with Morse index 2.

(III) If $\dim C(K) = 2$ *and* $(0,0)$ *is a boundary point of* $C(K)$*, we may assume by a simple rotation that* $p_+ = (0,r), p_- = (0,-r)$ *in* K *and for any* $p = (p_x, p_y) \in K \setminus \{p_+, p_-\}$*,* $p_x < 0$*. Then for* $\epsilon > 0$ *small* $\frac{1}{2}C_\lambda^l dist^2\big((x,y),K\big) \in C^{1,1}$ *and*

$$\frac{1}{2}C_\lambda^l dist^2\big((x,y),K\big) = \begin{cases} \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} - \frac{\lambda}{2}y^2 & (x,y) \in \mathbb{B}\big((0,0),\epsilon\big), x \leq 0 \\ \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}x^2 - \frac{\lambda}{2}y^2 & (x,y) \in \mathbb{B}\big((0,0),\epsilon\big), x > 0 \end{cases}$$

Consequently,

$$\lim_{\substack{x \to 0^- \\ y \to 0}} H(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}$$

with one negative eigenvalue and

$$\lim_{\substack{x \to 0^+ \\ y \to 0}} H(x,y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

with two negative eigenvalues and therefore $(0,0)$ *is a critical point with Morse index* $(1,2)$*. Furthermore, this critical point* $(0,0)$ *is an unstable equilibrium point (see Definition 3.2.5) for the system*

$$\dot{\underline{x}} = D\frac{1}{2}C_\lambda^l dist^2\big(\underline{x}, K\big).$$

Remark 3.2.8. Case (iii) happens due to the fact that $C_\lambda^l dist^2\big((x,y),K\big)$ is not C^2 near $(0,0)$ in this case.

Proof. *(I)* Suppose that $\dim C(K) = 1$. Since $(0,0) \in C(K)$, thus it is a critical point of $C_\lambda^l \mathrm{dist}^2\big((0,0),K\big)$ and $(0,0) \in M_K$ is defined by exactly two points. Let us consider these two points to be $(0,r)$ and $(0,-r)$, this means that K is a set of two points given as $K = \big\{(0,r),(0,-r)\big\}$ as shown in Figure 3.1 where we plot the convex hull $C\big((0,r),(0,-r)\big)$ and the ball of radius $r = \frac{1}{1+\lambda}$ centred at $(0,0)$ denoted by $\mathbb{S}^1\big((0,0),r\big)$.

We know from Lemma 3.1.2 that, for $(x,y) \in \mathbb{R}^2$,

$$C_\lambda^l \mathrm{dist}^2\big((x,y),K\big) = \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)\mathrm{dist}^2\Big((x,y),\frac{C(K)}{1+\lambda}\Big) - \lambda x^2 - \lambda y^2.$$

Therefore, for $(x,y) \in \mathbb{S}^1\big((0,0),\frac{r}{1+\lambda}\big)$ the lower transform is given by

$$\begin{aligned} C_\lambda^l \mathrm{dist}^2\big((x,y),K\big) &= \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)x^2 - \lambda x^2 - \lambda y^2 \\ &= \frac{\lambda}{1+\lambda}r^2 + x^2 - \lambda y^2. \end{aligned} \tag{3.2.15}$$

Hence $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is C^∞ near $(0,0)$ and the Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}.$$

The Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ and (3.2.15) both have just one negative eigenvalue. Therefore, the lower transform $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is a standard Morse function in a neighbourhood of point $(0,0)$ and the point $(0,0)$ is a standard (non-degenerate) critical point with Morse index 1.

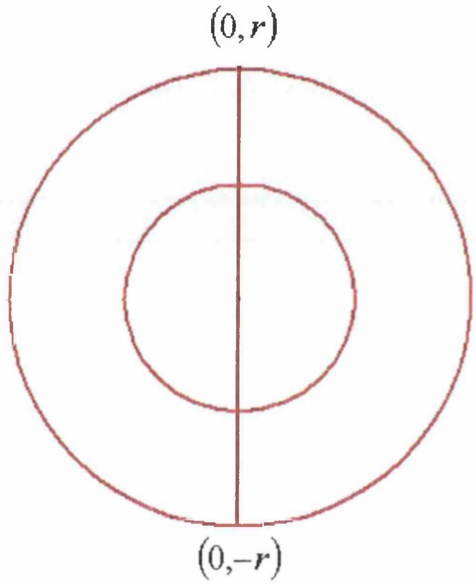

Figure 3.1: The convex hull $C\big((0,r),(0,-r)\big)$ and ball of radius $r = \frac{1}{1+\lambda}$ centred at $(0,0)$ denoted by $\mathbb{S}^1\big((0,0),\frac{1}{1+\lambda}\big)$.

(II) Suppose that $\dim C(K) = 2$ and $(0,0)$ is an interior point of $C(K)$. We know from Lemma 3.1.2 that, for $(x,y) \in \mathbb{R}^2$,

$$C^l_\lambda \mathrm{dist}^2\big((x,y),K\big) = \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)\mathrm{dist}^2\Big((x,y),\frac{C(K)}{1+\lambda}\Big) - \lambda x^2 - \lambda y^2$$

Suppose for $\epsilon > 0$ small enough that $\mathbb{S}^1\big((0,0),\epsilon\big) \subset \frac{C(K)}{1+\lambda}$ as shown in Figure 3.2.

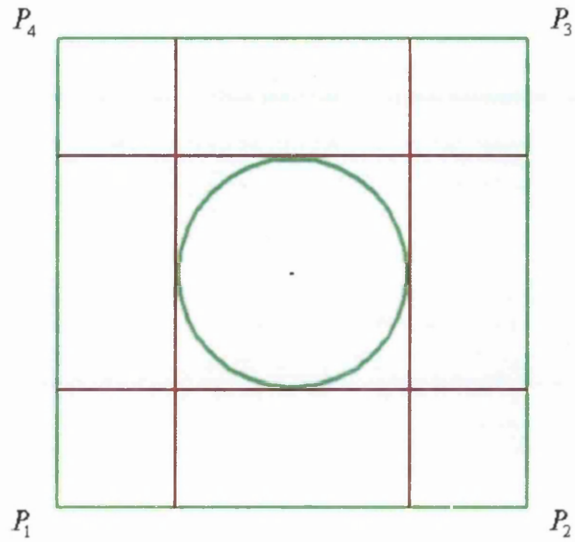

Figure 3.2: The convex hull $C(K)$ for $K = \{P_1, P_2, P_3, P_4\}$ and ball of radius $r = \epsilon$ centred at $(0,0)$ denoted by $\mathbb{S}^1\big((0,0),\epsilon\big)$.

Thus for (x,y) in the neighbourhood of $(0,0)$ the lower transform $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is

$$C^l_\lambda \mathrm{dist}^2\big((x,y),K\big) = \frac{\lambda}{1+\lambda}r^2 - \lambda x^2 - \lambda y^2 \tag{3.2.16}$$

Hence $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is C^∞ near $(0,0)$ and the Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is given by

$$H = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

The Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ and (3.2.16) both have two negative eigenvalues. Therefore, the lower transform $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is a standard Morse function in a neighbourhood of point $(0,0)$ and the point $(0,0)$ is a standard (non-degenerate) critical point with Morse index 2.

(III) Suppose that $\dim C(K) = 2$ and $(0,0)$ is a boundary point of $C(K)$. In the Figure 3.3 we plot the $C(K)$ for $K = \{P_1, P_2, P_3\}$ and ball of radius $\epsilon = \frac{1}{1+\lambda}$ centred at $(0,0)$ denoted by $\mathbb{S}^1_+\big((0,0),\epsilon\big)$ for $x \geq 0$ and $\mathbb{S}^1_-\big((0,0),\epsilon\big)$ for $x < 0$. We denote $\mathbb{S}^1_-\big((0,0),\epsilon\big) := \mathbb{S}^1\big((0,0),\epsilon\big) \cap \frac{C(K)}{1+\lambda}$ and $\mathbb{S}^1_+\big((0,0),\epsilon\big) := \mathbb{S}^1\big((0,0),\epsilon\big) \cap \left(\frac{C(K)}{1+\lambda}\right)^c$ and defined as

$$\mathbb{S}^1_-\big((0,0),\epsilon\big) = \Big\{(x,y) \in \mathbb{S}^1\big((0,0),\epsilon\big) : x \leq 0\Big\}$$

and

$$\mathbb{S}^1_+\big((0,0),\epsilon\big) = \big\{(x,y) \in \mathbb{S}^1\big((0,0),\epsilon\big) : x > 0\big\}.$$

Suppose that $(x,y) \in \mathbb{S}^1_-\big((0,0),\epsilon\big)$. Then the lower transform is

$$C^l_\lambda \mathrm{dist}^2\big((x,y),K\big) = \frac{\lambda}{1+\lambda} r^2 - \lambda x^2 - \lambda y^2. \tag{3.2.17}$$

Hence $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is C^∞ near $(0,0)$ and the Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is

$$\lim_{\substack{x\to 0^+ \\ y\to 0}} H(x,y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

The Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ and (3.2.17) both have two negative eigenvalues.

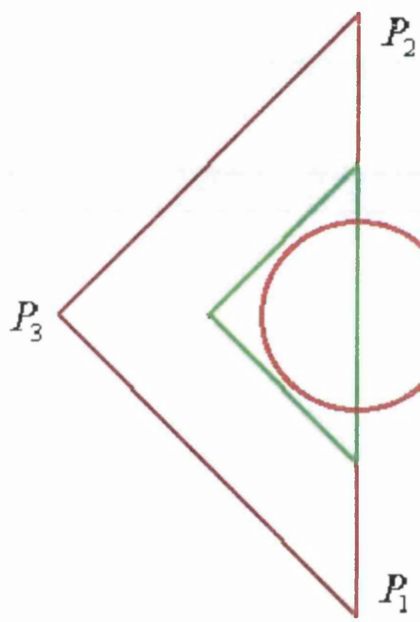

Figure 3.3: The convex hull $C(K)$ for $K = \{P_1, P_2, P_3\}$ and ball of radius $\epsilon = \frac{1}{1+\lambda}$ centred at $(0,0)$ denoted by $\mathbb{S}^1\big((0,0),\epsilon\big)$.

Now suppose that $(x,y) \in \mathbb{S}^1_+\big((0,0),\epsilon\big)$. Then the lower transform is

$$C^l_\lambda \mathrm{dist}^2\big((x,y),K\big) = \frac{\lambda}{1+\lambda} r^2 + x^2 - \lambda y^2. \tag{3.2.18}$$

The Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ is given by

$$\lim_{\substack{x\to 0^- \\ y\to 0}} H(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}.$$

The Hessian of $\frac{1}{2}C_\lambda^l \mathrm{dist}^2\big((x,y),K\big)$ and (3.2.18) both have only one negative eigenvalue. Since the critical point has regions with both one and two minus signs in its neightbourhood, we say that it is a critical point with Morse index $(1,2)$. Note that in this case, $C_\lambda^l \mathrm{dist}^2\big((x,y),K\big)$ is only $C^{1,1}$ near $(0,0)$ and not C^∞. □

Theorem 3.2.9 (Classification of Critical Points for $\mathbb{R}^3$). *Suppose $K \subset \mathbb{S}^2\big((0,0,0),r\big)$ is finite and $(0,0,0) \in C(K)$.*

(I) If $\dim C(K) = 1$, then $\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big)$ is C^∞ near $(0,0,0)$ and

$$\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big) = \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} + \frac{y^2}{2} - \frac{\lambda}{2}z^2$$

near $(0,0,0)$ and the critical point $(0,0,0)$ is a standard (non-degenerate) critical point with Morse index 1.

(II) If $\dim C(K) = 2$ and $(0,0,0)$ is a relative interior point of $C(K) \subset P_{yz}$. Then

$$\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big) = \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} - \frac{\lambda}{2}y^2 - \frac{\lambda}{2}z^2$$

near $(0,0,0)$ and the critical point $(0,0,0)$ is a standard (non-degenerate) critical point with Morse index 2.

(III) If $\dim C(K) = 2$ and $(0,0,0)$ is a boundary point of $C(K) \subset P_{yz}$. We may assume by a simple rotation that $p_+ = (0,0,r), p_- = (0,0,-r)$ in K and for any $p = (p_x,p_y,p_z) \in K \setminus \{p_+,p_-\}$, $p_y < 0$. Then $C_\lambda^l dist^2\big((x,y,z),K\big) \in C^{1,1}$ by [20, Theorem 3.1] and for $(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big)$ we have

$$\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big) = \begin{cases} \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} - \frac{\lambda}{2}y^2 - \frac{\lambda}{2}z^2 & y \le 0 \\ \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} + \frac{y^2}{2} - \frac{\lambda}{2}z^2 & y > 0 \end{cases}$$

Consequently,

$$\lim_{\substack{x\to 0\\ y\to 0^-\\ z\to 0}} H(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

with two negative eigenvalues and

$$\lim_{\substack{x\to 0\\ y\to 0^+\\ z\to 0}} H(x,y,z) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\lambda \end{pmatrix}$$

with one negative eigenvalue. Therefore the critical point $(0,0,0)$ *is called a critical point with Morse index* $(1,2)$. *Furthermore, this critical point* $(0,0,0)$ *is an unstable equilibrium point (see Definition 3.2.5) for the system*

$$\underline{\dot{x}} = D\frac{1}{2}C_\lambda^l dist^2\big(\underline{x}, K\big).$$

(IV) *If* $\dim C(K) = 3$ *and* $(0,0,0)$ *is an interior point of* $C(K)$. *Then* $\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big)$ *is* C^∞ *near* $(0,0,0)$ *and*

$$\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big) = \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}(x^2+y^2+z^2)$$

near $(0,0,0)$ *and the critical point* $(0,0,0)$ *is a standard (non-degenerate) critical point with Morse index* 3.

(V) *If* $\dim C(K) = 3$ *and* $(0,0,0)$ *is a point on a face of* $C(K)$, *say on the xy-plane* P_{xy}. *We may assume that if for any* $p = (p_x, p_y, p_z) \in K$, *then* $p_z \geq 0$. *Then* $C_\lambda^l dist^2\big((x,y,z),K\big) \in C^{1,1}$ *by [20, Theorem 3.1] and for* $(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big)$ *we have*

$$\frac{1}{2}C_\lambda^l dist^2\big((x,y,z),K\big) = \begin{cases} \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}(x^2+y^2+z^2) & z\geq 0\\ \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}(x^2+y^2) + \frac{z^2}{2} & z<0 \end{cases}$$

Consequently,

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^+}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & -\lambda \end{pmatrix}$$

with three negative eigenvalues and

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^-}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0\\ 0 & -\lambda & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

with two negative eigenvalues and therefore the critical point $(0,0,0)$ *is called a critical point with Morse index* $(2,3)$*. Furthermore, this critical point* $(0,0,0)$ *is an unstable equilibrium point (see Definition 3.2.5) for the system*

$$\underline{\dot{x}} = D\frac{1}{2}C^l_\lambda dist^2(\underline{x}, K).$$

(VI) If $\dim C(K) = 3$ *and* $(0,0,0)$ *is a point on an edge of* $C(K)$*. We may assume by a simple rotation that* $p_+ = (0,0,r), p_- = (0,0,-r), p_1 = (r,0,0), p_2 = (0,-r,0)$ *in* K *and for any* $p = (p_x, p_y, p_z) \in K \setminus \{p_+, p_-, p_1, p_2\}$ *and* $p_z \geq 0$*. Then* $C^l_\lambda dist^2\big((x,y,z), K\big) \in C^{1,1}$ *by [20, Theorem 3.1] and for* $(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big)$ *we have*

$$\frac{1}{2}C^l_\lambda dist^2\big((x,y,z),K\big) = \begin{cases} \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}(x^2+y^2+z^2) & x \geq 0, y \leq 0 \\ \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} - \frac{\lambda}{2}(y^2+z^2) & x \leq 0, y \leq 0 \\ \frac{\lambda}{2(1+\lambda)}r^2 + \frac{y^2}{2} - \frac{\lambda}{2}(x^2+z^2) & x \geq 0, y \geq 0 \\ \frac{\lambda}{2(1+\lambda)}r^2 + \frac{x^2}{2} + \frac{y^2}{2} - \frac{\lambda}{2}z^2 & x \leq 0, y \geq 0. \end{cases}$$

Consequently,

$$\lim_{\substack{x\to 0^+ \\ y \to 0^- \\ z \to 0}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

with three negative eigenvalues,

$$\lim_{\substack{x\to 0^- \\ y \to 0^- \\ z \to 0}} H(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

with two negative eigenvalues,

$$\lim_{\substack{x\to 0^+ \\ y \to 0^+ \\ z \to 0}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

with two negative eigenvalues and

$$\lim_{\substack{x \to 0^- \\ y \to 0^+ \\ z \to 0}} H(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

with one negative eigenvalue. Therefore the critical point $(0, 0, 0)$ *is called a critical point with Morse index* $(1, 2, 3)$. *Furthermore, this critical point* $(0, 0, 0)$ *is an unstable equilibrium point for the system*

$$\dot{\underline{x}} = D\frac{1}{2}C^l_\lambda dist^2(\underline{x}, K).$$

Remark 3.2.10. Case (III),(IV),(V) and (VI) happen due to the fact that $C^l_\lambda dist^2\big((x, y, z), K\big)$ is $C^{1,1}$ and not C^2.

Proof. (*I*) Suppose that $\dim C(K) = 1$ and $(0, 0, 0) \in C(K)$. Then $(0, 0, 0) \in M_K$ is realised by exactly two points of K. Let us consider these two points to be $p_+ = (0, 0, r)$ and $p_- = (0, 0, -r)$ in K, that is,

$$K = \big\{(0, 0, r), (0, 0, -r)\big\}.$$

We know from Lemma 3.1.2 that, for $(x, y, z) \in \mathbb{R}^3$,

$$C^l_\lambda \text{dist}^2\big((x, y, z), K\big) = \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)\text{dist}^2\Big((x, y, z), \frac{C(K)}{1+\lambda}\Big) - \lambda(x^2 + y^2 + z^2)$$

We suppose that $(x, y, z) \in \mathbb{S}^2\big((0, 0, 0), \frac{r}{1+\lambda}\big)$. Then

$$\begin{aligned} C^l_\lambda \text{dist}^2\big((x, y, z), K\big) &= \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)(x^2 + y^2) - \lambda(x^2 + y^2 + z^2) \\ &= \frac{\lambda}{(1+\lambda)}r^2 + x^2 + y^2 - \lambda z^2. \end{aligned} \tag{3.2.19}$$

Hence $C^l_\lambda \text{dist}^2\big((x, y, z), K\big)$ is C^∞ near $(0, 0, 0)$ and the Hessian of $\frac{1}{2}C^l_\lambda \text{dist}^2\big((x, y, z), K\big)$ is given by

$$H(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x,y,z)$ and (3.2.19) both have just one negative eigenvalue. Therefore, the lower transform $C^l_\lambda \text{dist}^2\big((x,y,z),K\big)$ is a standard Morse function in a neighbourhood of point $(0,0,0)$ and the critical point $(0,0,0)$ is a standard (non-degenerate) critical point with Morse index 1.

(II) Suppose that $\dim C(K) = 2$ and $(0,0,0)$ is a relative interior point of $C(K)$, and $C(K)$ is in yz-plane (P_{yz}) with $x=0$. We know from Lemma 3.1.2 that, for $(x,y,z) \in \mathbb{R}^3$,

$$C^l_\lambda \text{dist}^2\big((x,y,z),K\big) = \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)\text{dist}^2\Big((x,y,z),\frac{C(K)}{1+\lambda}\Big) - \lambda(x^2+y^2+z^2)$$

Suppose that for $\epsilon > 0$ small, $P_{yz} \cap \mathbb{S}^2\big((0,0,0),\epsilon\big) \subset \frac{C(K)}{1+\lambda}$. Then for (x,y,z) in a neighbourhood of $(0,0,0)$ the lower transform is of the form

$$C^l_\lambda \text{dist}^2\big((x,y,z),K\big) = \frac{\lambda}{1+\lambda}r^2 + x^2 - \lambda y^2 - \lambda z^2. \tag{3.2.20}$$

Hence $C^l_\lambda \text{dist}^2\big((x,y,z),K\big)$ is C^∞ near $(0,0,0)$ and the Hessian of $\frac{1}{2}C^l_\lambda \text{dist}^2\big((x,y,z),K\big)$ is given by

$$H(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x,y,z)$ and (3.2.20) both have two negative eigenvalues. Therefore, the lower transform $C^l_\lambda \text{dist}^2\big((x,y,z),K\big)$ is a standard Morse function in a neighbourhood of point $(0,0,0)$ and the critical point $(0,0,0)$ is a standard (non-degenerate) critical point with Morse index 2.

(III) Suppose that $\dim C(K) = 2$ and $(0,0,0)$ is a boundary point of $C(K)$. We know from Lemma 3.1.2 that, for $(x,y,z) \in \mathbb{R}^3$,

$$C^l_\lambda \text{dist}^2\big((x,y,z),K\big) = \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)\text{dist}^2\Big((x,y,z),\frac{C(K)}{1+\lambda}\Big) - \lambda(x^2+y^2+z^2)$$

Suppose that for $\epsilon > 0$ small and $y \le 0$, $P_{yz} \cap \mathbb{S}^2\big((0,0,0),\epsilon\big) \subset \frac{C(K)}{1+\lambda}$. Then we defined

$$\begin{aligned} \mathbb{S}^2_-\big((0,0,0),\epsilon\big) &:= P_{yz} \cap \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \frac{C(K)}{1+\lambda} = \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big) : y \le 0\big\}, \\ \mathbb{S}^2_+\big((0,0,0),\epsilon\big) &:= P_{yz} \cap \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c = \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big) : y > 0\big\}. \end{aligned}$$

First suppose that $(x, y, z) \in \mathbb{S}^2_-\big((0,0,0), \epsilon\big)$. Then the lower transform in this case is of the form

$$C^l_\lambda \mathrm{dist}^2\big((x, y, z), K\big) = \frac{\lambda}{1+\lambda} r^2 + x^2 - \lambda y^2 - \lambda z^2 \tag{3.2.21}$$

The Hessian of $\frac{1}{2} C^l_\lambda \mathrm{dist}^2\big((x, y, z), K\big)$ is given by

$$\lim_{\substack{x \to 0 \\ y \to 0^- \\ z \to 0}} H(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

Thus the Hessian $H(x, y, z)$ and (3.2.21) both have two negative eigenvalues.

Now suppose that $(x, y, z) \in \mathbb{S}^2_+\big((0,0,0), \epsilon\big)$. Then the lower transform in this case is of the form

$$C^l_\lambda \mathrm{dist}^2\big((x, y, z), K\big) = \frac{\lambda}{1+\lambda} r^2 + x^2 + y^2 - \lambda z^2. \tag{3.2.22}$$

The Hessian of $\frac{1}{2} C^l_\lambda \mathrm{dist}^2\big((x, y, z), K\big)$ is given by

$$\lim_{\substack{x \to 0 \\ y \to 0^+ \\ z \to 0}} H(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x, y, z)$ and (3.2.22) both have one negative eigenvalue. Thus in the neighbourhood of the critical point $(0, 0, 0)$ the lower transform and its Hessian have one and two negative eigenvalues. Therefore, we call it a critical point with Morse index $(1, 2)$. Further, the lower transform $C^l_\lambda \mathrm{dist}^2\big((x, y, z), K\big)$ is $C^{1,1}$ near $(0, 0, 0)$ and not C^∞.

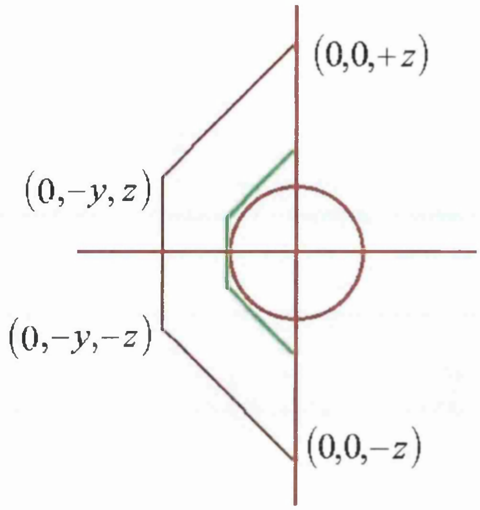

Figure 3.4: The YZ-Plane intersecting $\mathbb{S}^2_-\big((0,0,0),\epsilon\big)$ and $\mathbb{S}^2_+\big((0,0,0),\epsilon\big)$

(IV) Suppose that $\dim C(K) = 3$ and $(0,0,0)$ is an interior point of $C(K)$. We know from Lemma 3.1.2 that, for $(x,y,z) \in \mathbb{R}^3$,

$$C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big) = \frac{\lambda}{1+\lambda} r^2 + (1+\lambda)\mathrm{dist}^2\Big((x,y,z),\frac{C(K)}{1+\lambda}\Big) - \lambda(x^2+y^2+z^2)$$

We suppose that for $\epsilon > 0$ small, $\mathbb{S}^2\big((0,0,0),\epsilon\big) \subset \frac{C(K)}{1+\lambda}$. Then for (x,y,z) in the neighbourhood of $(0,0,0)$, we have

$$\frac{1}{2}C^l_\lambda dist^2\big((x,y,z),K\big) = \frac{\lambda}{2(1+\lambda)} r^2 - \frac{\lambda}{2}(x^2+y^2+z^2) \tag{3.2.23}$$

Hence $C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big)$ is C^∞ near $(0,0,0)$ and the Hessian of $\frac{1}{2}C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big)$ is given by

$$H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

This shows that the Hessian $H(x,y,z)$ and (3.2.23) both have three negative eigenvalues. Therefore, the lower transform $C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big)$ is a standard Morse function in a neighbourhood of point $(0,0,0)$ and the critical point $(0,0,0)$ is a standard (non-degenerate) critical point with Morse index 3.

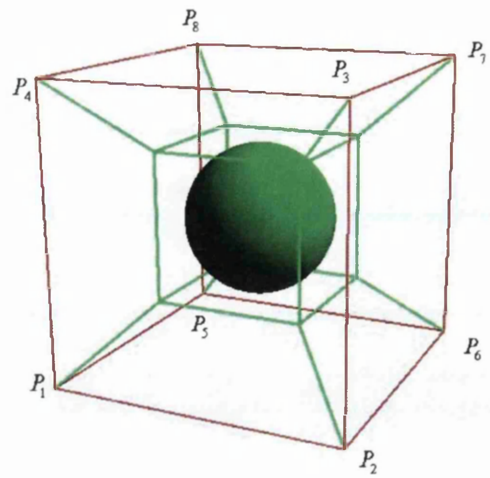

Figure 3.5: The critical point $(0,0,0)$ lies inside convex hull $C(K)$.

(V) Suppose $\dim C(K) = 3$ and $(0,0,0)$ is on a face of $C(K)$, say P_{xy}. We know from Lemma 3.1.2 that, for $(x,y,z) \in \mathbb{R}^3$,

$$C^l_\lambda \text{dist}^2\big((x,y,z),K\big) = \frac{\lambda}{1+\lambda}r^2 + (1+\lambda)\text{dist}^2\Big((x,y,z),\frac{C(K)}{1+\lambda}\Big) - \lambda(x^2+y^2+z^2)$$

Suppose that for $\epsilon > 0$ small, $P_{xy} \cap \mathbb{S}^2\big((0,0,0),\epsilon\big) \subset \frac{C(K)}{1+\lambda}$. Then we defined

$$\begin{aligned} \mathbb{S}^2_+\big((0,0,0),\epsilon\big) &:= \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \frac{C(K)}{1+\lambda} = \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big) : z \geq 0\big\}, \\ \mathbb{S}^2_-\big((0,0,0),\epsilon\big) &:= \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c = \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big) : z < 0\big\}. \end{aligned}$$

First suppose that $(x,y,z) \in \mathbb{S}^2_+\big((0,0,0),\epsilon\big)$. Then the lower transform in this case is

$$\frac{1}{2}C^l_\lambda dist^2\big((x,y,z),K\big) = \frac{\lambda}{2(1+\lambda)}r^2 - \frac{\lambda}{2}(x^2+y^2+z^2) \qquad (3.2.24)$$

The Hessian of $\frac{1}{2}C^l_\lambda \text{dist}^2\big((x,y,z),K\big)$ is given by

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^+}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x,y,z)$ and (3.2.24) both have three negative eigenvalues.

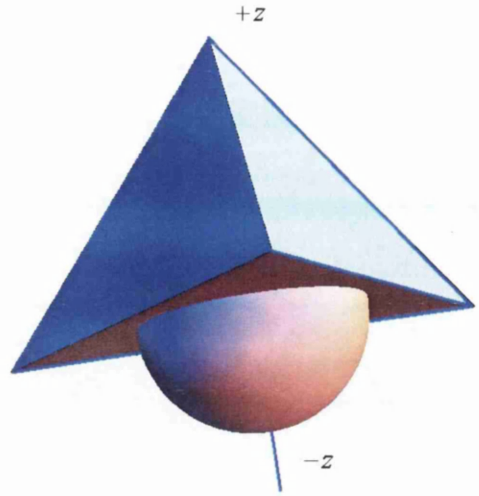

Figure 3.6: The critical point (0, 0, 0) lies on the face.

Now suppose that $(x, y, z) \in \mathbb{S}^2_-\big((0,0,0), \epsilon\big)$. Then the lower transform in this case is

$$C^l_\lambda \mathrm{dist}^2\big((x,y,z), K\big) = \frac{\lambda}{1+\lambda} r^2 - \lambda(x^2+y^2) + z^2. \tag{3.2.25}$$

The Hessian of $\frac{1}{2} C^l_\lambda \mathrm{dist}^2\big((x,y,z), K\big)$ is given by

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^-}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Hessian $H(x,y,z)$ and (3.2.25) both have two negative eigenvalues. Thus in the neighbourhood of the critical point $(0,0,0)$ the lower transform and its Hessian have two and three negative eigenvalues. Therefore, we call it a critical point with Morse index $(2, 3)$. Further, the $C^l_\lambda \mathrm{dist}^2\big((x,y,z), K\big)$ is $C^{1,1}$ near $(0,0,0)$ and not C^∞.

(VI) Suppose that $(0,0,0)$ is on an edge of $C(K)$. We define the following

$$\begin{aligned}
\mathbb{S}^2_1 &:= \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) \cap \frac{C(K)}{1+\lambda} &&= \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) : x \geq 0, y \leq 0\big\},\\
\mathbb{S}^2_2 &:= \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c &&= \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) : x \geq 0, y \geq 0\big\},\\
\mathbb{S}^2_3 &:= \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c &&= \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) : x \leq 0, y \geq 0\big\},\\
\mathbb{S}^2_4 &:= \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c &&= \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0), \tfrac{r}{1+\lambda}\big) : x \leq 0, y \leq 0\big\}
\end{aligned}$$

Suppose $(x, y, z) \in \mathbb{S}_1^2$, then we have

$$\frac{1}{2} C_\lambda^l dist^2\big((x, y, z), K\big) = \frac{\lambda}{2(1+\lambda)} r^2 - \frac{\lambda}{2}(x^2 + y^2 + z^2). \tag{3.2.26}$$

The Hessian of $\frac{1}{2} C_\lambda^l \mathrm{dist}^2\big((x, y, z), K\big)$ is given by

$$\lim_{\substack{x \to 0^+ \\ y \to 0^- \\ z \to 0}} H(x, y, z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x, y, z)$ and (3.2.26) both have three negative eigenvalues.

Suppose $(x, y, z) \in \mathbb{S}_2^2$, then we have

$$\frac{1}{2} C_\lambda^l dist^2\big((x, y, z), K\big) = \frac{\lambda}{2(1+\lambda)} r^2 + \frac{y^2}{2} - \frac{\lambda}{2}(x^2 + z^2) \tag{3.2.27}$$

The Hessian of $\frac{1}{2} C_\lambda^l \mathrm{dist}^2\big((x, y, z), K\big)$ is given by

$$\lim_{\substack{x \to 0^+ \\ y \to 0^+ \\ z \to 0}} H(x, y, z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x, y, z)$ and (3.2.27) both have two negative eigenvalues.

Suppose $(x, y, z) \in \mathbb{S}_3^2$, then we have

$$\frac{1}{2} C_\lambda^l dist^2\big((x, y, z), K\big) = \frac{\lambda}{2(1+\lambda)} r^2 + \frac{x^2}{2} + \frac{y^2}{2} - \frac{\lambda}{2} z^2 \tag{3.2.28}$$

The Hessian of $\frac{1}{2} C_\lambda^l \mathrm{dist}^2\big((x, y, z), K\big)$ is given by

$$\lim_{\substack{x \to 0^- \\ y \to 0^+ \\ z \to 0}} H(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x, y, z)$ and (3.2.28) both have one negative eigenvalues.

Suppose $(x, y, z) \in \mathbb{S}_4^2$, then we have

$$\frac{1}{2} C_\lambda^l dist^2\big((x, y, z), K\big) = \frac{\lambda}{2(1+\lambda)} r^2 + \frac{x^2}{2} - \frac{\lambda}{2}(y^2 + z^2) \tag{3.2.29}$$

The Hessian of $\frac{1}{2}C_\lambda^l \mathrm{dist}^2\big((x, y, z), K\big)$ is given by

$$\lim_{\substack{x\to 0^- \\ y\to 0^- \\ z\to 0}} H(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

The Hessian $H(x, y, z)$ and (3.2.29) both have two negative eigenvalues. Therefore, $(0, 0, 0)$ is a critical point with Morse index $(1, 2, 3)$.

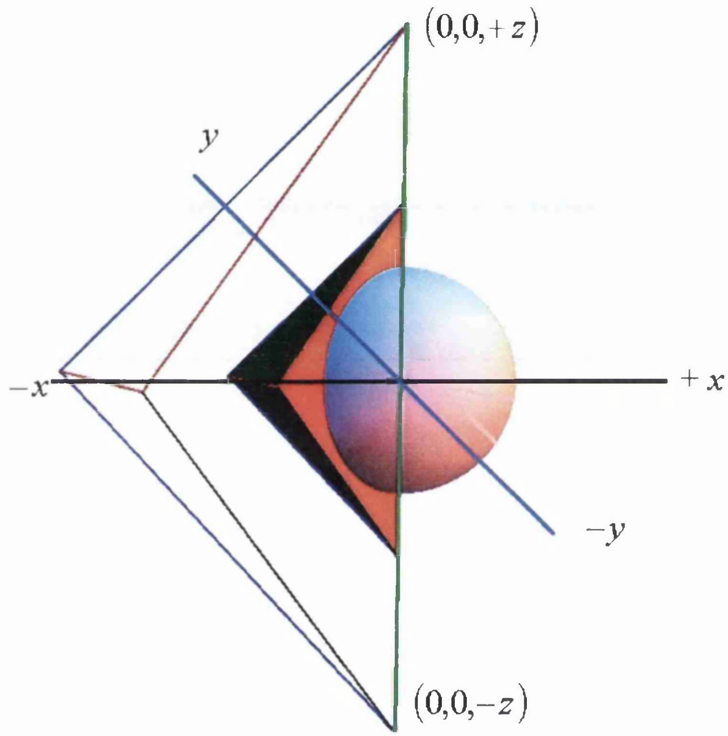

Figure 3.7: The critical point $(0, 0, 0)$ lies on an edge.

$\square$

3.3 Prototype Examples of Morse Indices of Critical Points of the Lower Transforms

We illustrate the three typical situations of Morse indices of the critical points, that is, for a smoothed squared distance function the critical points has Morse index 1, 2 in two dimensions and index 1,2 and 3 in three dimensions. We will examine the Morse indices of non-degenerate and degenerate critical points in $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.3.1 Morse Indices of Non-degenerate Critical Points

We examine Morse indices for the non-degenerate critical points with the help of some particular examples. The first example shows the situation in which the critical point $x_0 = (0,0)$ is in the interior of the $C(K)$ where $\dim C(K(x_0)) = 1$. Then in this case the critical point will be a standard (non-degenerate) critical point with Morse index 1. In the second example we illustrate the situation in which the critical point $x_0 = (0,0)$ is in the interior of $C(K)$ and $\dim C(K(x_0)) = 2$. Then this shows that the critical point in this case is a standard (non-degenerate) critical point but with Morse index 2. Finally we illustrate the situation in which the critical point $x_0 = (0,0,0)$ is in the interior of $C(K)$ and $\dim C(K(x_0)) = 3$. Then this shows that the critical point in this case is a standard (non-degenerate) critical point with Morse index 3.

Example 3.3.1. We consider the lower transform $h(x,y) := C_\lambda^l \mathrm{dist}^2\big((x,y),K\big)$ of the squared distance function to the finite set $K = \big\{(0,1),(0,-1)\big\}$ given by

$$h(x,y) = \begin{cases} x^2 + (y-1)^2 & y > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + x^2 - \lambda|y|^2 & |y| \le \frac{1}{1+\lambda} \\ x^2 + (y+1)^2 & y < -\frac{1}{1+\lambda}. \end{cases}$$

Note that $C\big(K(0,0)\big)$ is the straight line $\{(x,y) \in \mathbb{R}^2 : (x,y) = \big(0, \lambda - (1-\lambda)\big), 0 \le \lambda \le 1\}$. So clearly $(0,0) \in C\big(K(0,0)\big)$, and $\dim C\big(K(0,0)\big) = 1$.

In the Figure 3.8 we plot the $C\big(\{P_+, P_-\}\big)$ and ball of radius $r = 1$ and $r = \frac{1}{1+\lambda}$ centred at $(0,0)$ denoted by $\mathbb{S}^1\big((0,0),r\big)$. Then the lower transform $h(x,y) \in C^\infty$ and for $(x,y) \in \mathbb{S}^1\big((0,0),\frac{1}{1+\lambda}\big)$ we have

$$h(x,y) = \frac{\lambda}{1+\lambda} + x^2 - \lambda|y|^2. \tag{3.3.30}$$

The Hessian of $\frac{1}{2}h(x, y)$ at $(0, 0)$ is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Thus the Hessian and (3.3.30) both have only one negative eigenvalue. Therefore, $(0, 0)$ is a standard (non-degenerate) critical point with Morse index 1.

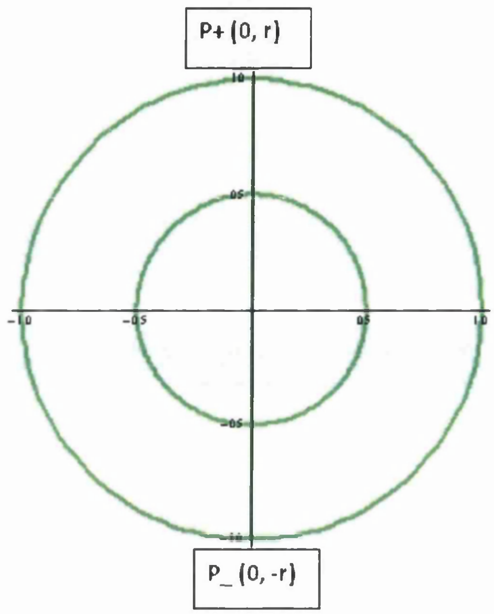

Figure 3.8: The convex hull $C(\{P_+, P_-\})$ and ball of radius $r = 1$ and $r = \frac{1}{1+\lambda}$ centred at $(0, 0)$ denoted by $\mathbb{S}^1\big((0, 0), r\big)$.

Example 3.3.2. We consider the lower transform $h(x, y) := C_\lambda^l \text{dist}^2\big((x, y), K\big)$ for a set K of the type in Example (4.2.3) in Chapter 4. Note that $C\big(K(0, 0)\big)$ is a large triangle in the plane and the region inside, so clearly $(0, 0) \in C\big(K(0, 0)\big)$ and $\dim C\big(K(0, 0)\big) = 2$. Suppose $\epsilon > 0$ small enough such that $\mathbb{S}^1\big((0, 0), \epsilon\big) \subset C\Big(\frac{K(0,0)}{1+\lambda}\Big)$. Let for $(x, y) \in \mathbb{S}^1\big((0, 0), \epsilon\big)$ we find the lower transform in the neighbourhood of $(0, 0)$. In the Figure 3.9 we plot the $C\big(\{P_1, P_2, P_3\}\big)$ and ball of radius $r = 1$ and $r = \epsilon$ centred at $(0, 0)$. Then the formula for the lower transform $h(x, y) \subset C^2$ is given by

$$h(x, y) = \frac{\lambda}{1 + \lambda} - \lambda x^2 - \lambda y^2. \tag{3.3.31}$$

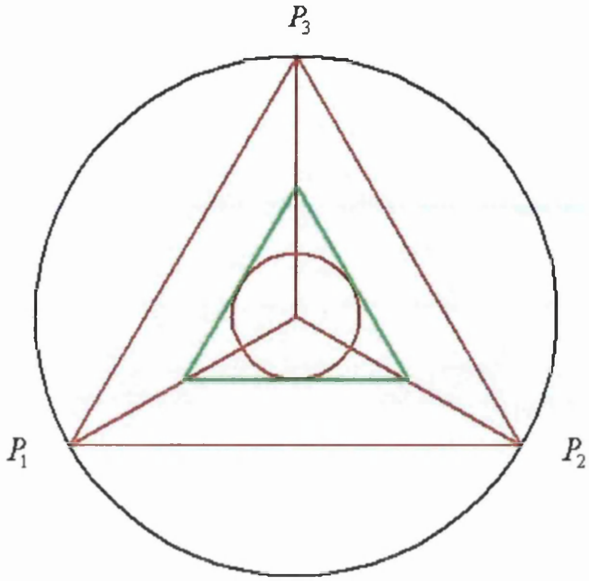

Figure 3.9: The convex hull $C(\{P_1, P_2, P_3\})$ and ball of radius $r = 1$ and $r = \epsilon$ centred at $(0, 0)$ denoted by $\mathbb{S}^1((0, 0), r)$.

The Hessian of $\frac{1}{2}h(x, y)$ at $(0, 0)$ is given by

$$H = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Thus the Hessian and (3.3.31) both have two negative eigenvalues. Therefore, $(0, 0)$ in this case is a standard (non-degenerate) critical point with Morse index 2.

Example 3.3.3. We consider the lower transform $h(x, y, z) := C_\lambda^l \mathrm{dist}^2((x, y, z), K)$ of the squared distance function to finite set K of the type in Example (4.2.4) in Chapter 4. Note that $C(K(0, 0, 0))$ is a large cube in $\mathbb{R}^3$, so clearly $(0, 0, 0) \in C(K(0, 0, 0))$ and $\dim C(K(0, 0, 0)) = 3$. Suppose $\epsilon > 0$ small enough such that $\mathbb{S}^2((0, 0, 0), \epsilon) \subset C\left(\frac{K(0,0,0)}{1+\lambda}\right)$. In the Figure 3.10 we plot $C(\{P_1, \ldots, P_8\})$ and ball of radius $r = \epsilon$ centred at $(0, 0, 0)$. Then the formula for the lower transform $h(x, y, z) \subset C^2$ is given by

$$h(x, y, z) = \frac{3\lambda}{1 + \lambda} - \lambda(x^2 + y^2 + z^2) \tag{3.3.32}$$

The Hessian of $\frac{1}{2}h(x,y,z)$ at $(0,0,0)$ is given by

$$H = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

Thus the Hessian and (3.3.32) both have three negative eigenvalues. Therefore, $(0,0,0)$ is a standard (non-degenerate) critical point with Morse index 3.

3.3.2 Morse Indices of Degenerate Critical Points

In this section we will examine the degenerate critical point with Morse indices classified in Theorem 3.2.9 with the help of particular examples. In first example, we examine the situation for which a critical point $x_0 = (0,0)$ is a boundary point of $C(K(x_0))$ and $\dim C(K(x_0)) = 2$, then in this case the critical point is an index (1,2) degenerate critical point.

Example 3.3.4. We consider $h(x,y) := C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ the lower transform of squared distance function to finite set $K = K\big((0,0)\big)$ from Example (4.2.2) and suppose $\epsilon := \frac{1}{\sqrt{2(1+\lambda)}}$. Then we define the following:

$$\begin{aligned} \mathbb{S}^1_+\big((0,0),\epsilon\big) &:= \mathbb{S}^1\big((0,0),\epsilon\big) \cap \frac{C(K)}{1+\lambda} = \big\{(x,y) \in \mathbb{S}^1\big((0,0),\epsilon\big) : y \geq 0\big\} \\ \mathbb{S}^1_-\big((0,0),\epsilon\big) &:= \mathbb{S}^1\big((0,0),\epsilon\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c = \big\{(x,y) \in \mathbb{S}^1\big((0,0),\epsilon\big) : y < 0\big\} \end{aligned}$$

Then the formula for the lower transform $h(x,y)$ for each $(x,y) \in \mathbb{S}^1_+\big((0,0),\epsilon\big)$, that is, in the neighbourhood of $(0,0)$ is given by

$$\frac{1}{2}h(x,y) = \frac{1}{2}\big(\frac{\lambda}{1+\lambda} - \lambda x^2 - \lambda y^2\big) \tag{3.3.33}$$

and its Hessian at $(0,0)$ is

$$\lim_{\substack{x\to 0 \\ y \to 0^+}} H(x,y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

Similarly, for each $(x,y) \in \mathbb{S}^1_-\big((0,0),\epsilon\big)$, that is, in the neighbourhood of $(0,0)$ is given by

$$\frac{1}{2}h(x,y) = \frac{1}{2}\big(\frac{\lambda}{1+\lambda} - \lambda x^2 + y^2\big) \tag{3.3.34}$$

and its Hessian at $(0,0)$ is

$$\lim_{\substack{x\to 0\\ y\to 0^-}} H(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}.$$

We can notice that (3.3.33) for $(x,y) \in \mathbb{S}^1_+\big((0,0),\epsilon\big)$ and its Hessian at $(0,0)$ has two negative eigenvalues whereas (3.3.34) for $(x,y) \in \mathbb{S}^1_-\big((0,0),\epsilon\big)$ and its Hessian has only one negative eigenvalue. Therefore, we call $(0,0)$ an unstable equilibrium critical point with Morse index $(1,2)$ of the system 3.2.14.

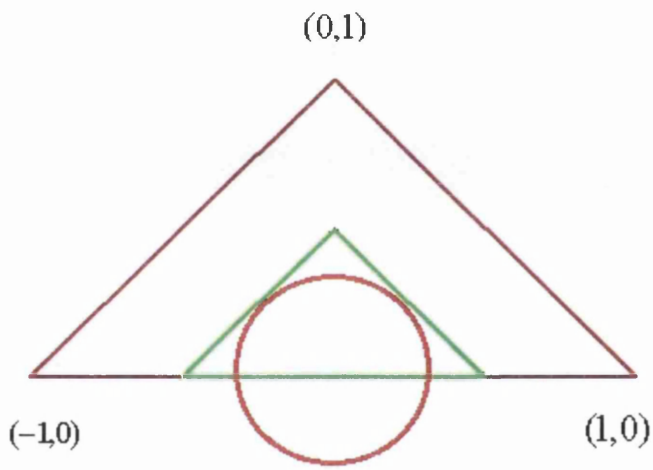

Figure 3.10: The convex hull $C\big(\{(-1,0),(1,0),(0,1)\}\big)$ and ball of radius $r = \epsilon$ centred at $(0,0)$ denoted by $\mathbb{S}^1\big((0,0),\epsilon\big)$.

In this example we illustrate that if critical point $x_0 = (0,0,0)$ lies on the interior of a face of $C(K(x_0))$ and $\dim C(K(x_0)) = 3$, then we call x_0 a degenerate critical point with Morse index (2,3).

Example 3.3.5. We consider $h(x,y,z) := C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big)$ the lower transform of squared distance function to finite set $K = K\big((0,0,0)\big)$ from Example 4.2.6 and suppose

$\epsilon := \frac{1}{2(1+\lambda)}$. Then we define the following:

$$\begin{aligned}
\mathbb{S}^2_+\big((0,0,0),\epsilon\big) &:= \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \frac{C(K)}{1+\lambda} = \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big) : z \geq 0\big\} \\
\mathbb{S}^2_-\big((0,0,0),\epsilon\big) &:= \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c = \big\{(x,y,z) \in \mathbb{S}^2\big((0,0,0),\epsilon\big) : z < 0\big\}
\end{aligned}$$

Then the formula for the lower transform $h(x,y,z)$ for each $(x,y,z) \in \mathbb{S}^2_+\big((0,0,0),\epsilon\big)$, that is, in the neighbourhood of $(0,0,0)$ is given by

$$\frac{1}{2}h(x,y,z) = \frac{1}{2}\Big(\frac{\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2)\Big) \tag{3.3.35}$$

and its Hessian at $(0,0,0)$ is

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^+}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

Similarly, for each $(x,y,z) \in \mathbb{S}^2_-\big((0,0,0),\epsilon\big)$, that is, in the neighbourhood of $(0,0,0)$ is given by

$$\frac{1}{2}h(x,y,z) = \frac{1}{2}\Big(\frac{\lambda}{1+\lambda} - \lambda(x^2+y^2) + z^2\Big) \tag{3.3.36}$$

and its Hessian at $(0,0,0)$ is

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^-}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We notice that (3.3.35) for $(x,y,z) \in \mathbb{S}^2_+\big((0,0,0),\epsilon\big)$ and its Hessian at $(0,0,0)$ has three negative eigenvalues whereas (3.3.36) for $(x,y,z) \in \mathbb{S}^2_-\big((0,0,0),\epsilon\big)$ and its Hessian at $(0,0,0)$ has two negative eigenvalues. Therefore, we call $(0,0,0)$ an unstable equilibrium critical point with Morse index $(2,3)$ of the system 3.2.14.

In this example we explain the situation when the critical point $x_0 = (0,0,0)$ lies on an edge (relative boundary of $C(K(x_0))$) and $\dim C(K(x_0)) = 3$. In this case, the critical point x_0 is an index (1,2,3) degenerate critical point.

Example 3.3.6. We consider $h(x,y,z) := C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big)$ the lower transform of squared distance function to finite set K from Example 4.2.5 and suppose $\epsilon := \frac{1}{\sqrt{2}(1+\lambda)}$. Then we define the following:

$$\begin{aligned}
\mathbb{S}^2_+\big((0,0,0),\epsilon\big) &:= \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \frac{C(K)}{1+\lambda} \\
\mathbb{S}^2_-\big((0,0,0),\epsilon\big) &:= \mathbb{S}^2\big((0,0,0),\epsilon\big) \cap \Big(\frac{C(K)}{1+\lambda}\Big)^c
\end{aligned}$$

Then the formula for the lower transform $h(x,y,z)$ for each $(x,y,z) \in \mathbb{S}^2_+\big((0,0,0),\epsilon\big)$, that is, in the neighbourhood of $(0,0,0)$ is given by

$$\frac{1}{2}h(x,y,z) = \frac{1}{2}\big(\frac{\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2)\big) \tag{3.3.37}$$

and its Hessian at $(0,0,0)$ is

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0^+}} H(x,y,z) = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

Then the Hessian $H(x,y,z)$ and (3.3.37) both have three negative eigenvalues.
The lower transform $h(x,y,z)$ for each $(x,y,z) \in \mathbb{S}^2_-\big((0,0,0),\epsilon\big)$, that is, in the neighbourhood of $(0,0,0)$ is given by

$$\frac{1}{2}h(x,y,z) = \frac{1}{2}\begin{cases} \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}(-x+\sqrt{3}y)^2 - \lambda(x^2+y^2+z^2) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}(x+\sqrt{3}y)^2 - \lambda(x^2+y^2+z^2) \\ \frac{\lambda}{1+\lambda} + x^2+y^2-\lambda z^2 \end{cases} \tag{3.3.38}$$

From the Hessian of $\frac{1}{2}h(x,y,z)$ at $(0,0,0)$ for first region,

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} H(x,y,z) = \begin{pmatrix} \frac{1-3\lambda}{4} & -\frac{\sqrt{3}(1+\lambda)}{4} & 0 \\ -\frac{\sqrt{3}(1+\lambda)}{4} & \frac{3-\lambda}{4} & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

we note that there are two negative eigenvalues as $(1-3\lambda)(3-\lambda)-3(1+\lambda)^2 = -16\lambda < 0$.
Similarly, the Hessian of $\frac{1}{2}h(x,y,z)$ at $(0,0,0)$ for second region,

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} H(x,y,z) = \begin{pmatrix} \frac{1-3\lambda}{4} & \frac{\sqrt{3}(1+\lambda)}{4} & 0 \\ \frac{\sqrt{3}(1+\lambda)}{4} & \frac{3-\lambda}{4} & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

has two nagative eigenvalues. Finally the Hessian of $\frac{1}{2}h(x,y,z)$ at $(0,0,0)$ for last region,

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} H(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

has one negative eigenvalue. Thus we note that (3.3.37) for $(x,y,z) \in \mathbb{S}^2_+\big((0,0,0),\epsilon\big)$ and its Hessian at $(0,0,0)$ has three negative eigenvalues. On the other hand, in (3.3.38) for $(x,y,z) \in \mathbb{S}^2_-\big((0,0,0),\epsilon\big)$ and its Hessian at $(0,0,0)$ has one and two both negative eigenvalues for $\lambda > 0$. Therefore, we call $(0,0,0)$ an unstable equilibrium critical point with Morse index $(1,2,3)$ of the system 3.2.14.

Chapter 4

Descriptive Properties of $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ to finite set $K \subset \mathbb{R}^n$

In this chapter we study the local behaviour of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ for the squared-distance function to a finite set $K \subset \mathbb{R}^n$. We show some estimations and properties of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ in the neighbourhood of singular points for large $\lambda > 0$. On the basis of these local estimations and local properties we prove that the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ has a semi-global property. We also present a result that under certain conditions, $C^l_\lambda \mathrm{dist}^2(x, K)$ can represented in each simplex in $C(K)$ of a triangulation of $C(K)$. The motivation for studying local properties of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ for the squared distance function is to show how the lower transform modifies the original squared distance function near the medial axis when λ is large enough. This gives an idea of how the gradient flow defined by $\dot{x} = \frac{1}{2} D C^l_\lambda \mathrm{dist}^2(x, K)$ will move near the singular points of the squared distance function. This gradient flow will be explained, with a view to applications of surface reconstruction, in Chapter 5. We also establish a global result for triangulation simplices based on the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ to a finite set $K \subset \mathbb{R}^n$. This result show that for all triangulated simplices of the set K, the lower transform is the same as that for a single triangulated simplex. Furthermore, we calculate the explicit formulae of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ for the squared distance functions to finite sets for some concrete examples of K in one, two and three dimensions.

4.1 Local properties of the lower transform

In this section we will be mainly discussing the local representation of the lower transform $C_\lambda^l \mathrm{dist}^2(x, K)$ for the squared-distance function to a finite set $K \subset \mathbb{R}^n$. Throughout this section, for every x in $\mathbb{R}^n$ we will consider $x \in M_K \setminus K$ where $K \subset \mathbb{R}^n$ a finite set, $K(x) \subseteq K$ a finite subset of K (see Definition 2.1.4), and M_K the medial axis of K (see Definition 2.1.3). Furthermore, we consider the closed ball denoted by $\bar{B}(x, r(x))$ of radius $r(x) > 0$ centred at x (see Definition 2.1.2) and radius $r(x)$ is defined as $r(x) := dist(x, K(x))$. We start with an estimation of distance between a local point $x_i \in \mathbb{R}^n \setminus K$ and $x \in M_K$ and show that for $\lambda > 4$ the estimate is given by $|x_i - x| \le \frac{6}{\sqrt{\lambda}} \mathrm{dist}(x, K)$ if x_i's are involved in the representation of the lower transform $C_\lambda^l \mathrm{dist}^2(x, K)$ for the squared distance function to the finite set K.

Theorem 4.1.1 (Locality Theorem). *Let $K \subset \mathbb{R}^n$ be a finite set, $x \in \mathbb{R}^n$. Suppose $x_1, ..., x_k \in \mathbb{R}^n$, $2 \le k \le n+1$ be such that*

$$C_\lambda^l dist^2(x, K) = \sum_{i=1}^k \tau_i \big(dist^2(x_i, K) + \lambda |x_i - x|^2 \big) \tag{4.1.1}$$

for which $\tau_1, ..., \tau_k > 0, \sum_{i=1}^k \tau_i = 1, \sum_{i=1}^k \tau_i x_i = x$. Then for $\lambda > 4$,

$$|x_i - x| \le \frac{6}{\sqrt{\lambda}}\, dist(x, K). \tag{4.1.2}$$

Consequently,

$$\begin{aligned} C_\lambda^l dist^2(x, K) = \quad & \inf \Big\{ \sum_{i=1}^{n+1} \tau_i \big(dist^2(y_i, K) + \lambda |y_i - x|^2 \big) : \tau_i \ge 0, \sum_{i=1}^{n+1} \tau_i = 1, \sum_{i=1}^{n+1} \tau_i y_i = x, \\ & y_i \in \bar{B}\big(x, r_\lambda(x)\big), i = 1, ..., k \Big\} \end{aligned}$$

where $r_\lambda(x) = \frac{6}{\sqrt{\lambda}}\, dist(x, K)$.

Proof. Without loss of generality assume that $x = 0$. Thus we need to show that $|x_i| \leq \frac{6}{\sqrt{\lambda}}\,\mathrm{dist}(0, K)$. We know from (4.1.1) that for $x = 0$

$$C_\lambda^l \mathrm{dist}^2(0, K) = \sum_{i=1}^{k} \tau_i\big(\mathrm{dist}^2(x_i, K) + \lambda|x_i|^2\big) \tag{4.1.3}$$

for $\tau_1, ..., \tau_k > 0$, $\sum_{i=1}^k \tau_i = 1$ and $\sum_{i=1}^k \tau_i x_i = 0$. We also suppose the supporting plane (an affine function) l given by

$$l(x) = 2a \cdot x + b = b\big|_{x=0}\,, \qquad \text{where } a \text{ is a vector and } b \in \mathbb{R} \tag{4.1.4}$$

such that it satisfies the following two conditions

(a) $l(x) \leq \mathrm{dist}^2(x, K) + \lambda|x|^2, \qquad \forall\, x \in \mathbb{R}^n$

(b) $l(x_i) = \mathrm{dist}^2(x_i, K) + \lambda|x_i|^2, \qquad \mathit{for}\ i = 1, 2, ..., k.$

Since the supporting plane satisfies $l(x_i) = 2a \cdot x_i + b$, hence from (b) and (4.1.4) , there exists $x_i' \in K$ such that

$$2a \cdot x_i + b = |x_i - x_i'|^2 + \lambda|x_i|^2 \tag{4.1.5}$$

Now (see differentiability Lemma 2.2.20) the equation (4.1.5) implies that

$$\begin{aligned} 2a &= 2(x_i - x_i') + 2\lambda x_i \\ |x_i - x_i'|^2 &= \lambda^2\Big|x_i - \frac{a}{\lambda}\Big|^2 \end{aligned} \tag{4.1.6}$$

Completing the square in (4.1.5) gives,

$$\frac{|a|^2}{\lambda} + b = |x_i - x_i'|^2 + \lambda\Big|x_i - \frac{a}{\lambda}\Big|^2. \tag{4.1.7}$$

Again by substituting the value of $|x_i - x_i'|^2$ from (4.1.6) into (4.1.7) and thus simplifies to

$$\Big|x_i - \frac{a}{\lambda}\Big|^2 = \frac{|a|^2}{\lambda^2(1+\lambda)} + \frac{b}{\lambda(1+\lambda)}. \tag{4.1.8}$$

and taking the square root gives

$$\Big|x_i - \frac{a}{\lambda}\Big| = \sqrt{\frac{|a|^2}{\lambda^2(1+\lambda)} + \frac{b}{\lambda(1+\lambda)}} \leq \frac{|a|}{\lambda\sqrt{(1+\lambda)}} + \sqrt{\frac{b}{\lambda(1+\lambda)}}. \tag{4.1.9}$$

Now from (a) and (4.1.4) we have

$$2a \cdot x + b \leq \mathrm{dist}^2(x, K) + \lambda |x|^2.$$

Let $x = \frac{a}{\lambda}$, then it follows that

$$2\frac{|a|^2}{\lambda} + b \leq \mathrm{dist}^2\big(\frac{a}{\lambda}, K\big) + \frac{|a|^2}{\lambda}.$$

Since $b > 0$, some simplification gives

$$\frac{|a|^2}{\lambda} \leq \mathrm{dist}^2\big(\frac{a}{\lambda}, K\big) \leq \big|\frac{a}{\lambda} - y\big|^2, \qquad \forall\ y \in K.$$

Taking the square root and using the triangle inequality, it follows that

$$\frac{|a|}{\sqrt{\lambda}} \leq \big|\frac{a}{\lambda} - y\big| \leq \big|\frac{a}{\lambda}\big| + |y|. \tag{4.1.10}$$

If we take $y \in K$ such that $\mathrm{dist}(0, K) = |y - 0|$, then (4.1.10) can be rewritten as

$$\frac{|a|}{\sqrt{\lambda}} \leq \big|\frac{a}{\lambda}\big| + \mathrm{dist}(0, K)$$

and simplification gives

$$|a| \leq \frac{\lambda}{\sqrt{\lambda} - 1} \mathrm{dist}(0, K) \tag{4.1.11}$$

We know from Chapter 3 Lemma 3.1.2 that at $x = 0$, $C^l_\lambda \mathrm{dist}^2(0, K) = b$ and so from (4.1.4), $C^l_\lambda \mathrm{dist}^2(0, K) = l(x)\big|_{x=0}$. Thus again from differentiablity Lemma 2.2.20 and simplification using (4.1.11) we have

$$\begin{aligned} DC^l_\lambda \mathrm{dist}^2(0, K) &= D(2ax + b) \\ \big|DC^l_\lambda \mathrm{dist}^2(0, K)\big| &\leq \frac{2\lambda}{\sqrt{\lambda} - 1} \mathrm{dist}(0, K) \end{aligned}$$

We know $b = l(x)\big|_{x=0} = C^l_\lambda \mathrm{dist}^2(0, K) \leq \mathrm{dist}^2(0, K)$, which implies that $\sqrt{b} \leq \mathrm{dist}(0, K)$. Therefore equation (4.1.9) implies that

$$\big|x_i - \frac{a}{\lambda}\big| \leq \frac{|a|}{\lambda\sqrt{(1+\lambda)}} + \frac{1}{\sqrt{\lambda(1+\lambda)}} \mathrm{dist}(0, K). \tag{4.1.12}$$

We know from triangle inequality that

$$\big|x_i - \frac{a}{\lambda}\big| \geq |x_i| - \frac{|a|}{\lambda}.$$

Hence (4.1.12) can be written as

$$|x_i| \leq \frac{|a|}{\lambda} + \frac{|a|}{\lambda\sqrt{(1+\lambda)}} + \frac{1}{\sqrt{\lambda(1+\lambda)}}\mathrm{dist}(0,K)$$

Now substituting the value of $|a|$ from (4.1.11), we get

$$|x_i| \leq \Big(\frac{1}{\sqrt{\lambda}-1} + \frac{1}{\sqrt{1+\lambda}\big(\sqrt{\lambda}-1\big)} + \frac{1}{\sqrt{\lambda(1+\lambda)}}\Big)\mathrm{dist}(0,K).$$

We estimate for $\lambda > 1$ that

$$|x_i| \leq \frac{3}{\sqrt{\lambda}-1}\mathrm{dist}(0,K),$$

and for $\lambda \geq 4$

$$|x_i| \leq \frac{6}{\sqrt{\lambda}}\mathrm{dist}(0,K),$$

and hence the result follows. □

In the following Proposition 4.1.2, we prove that for any point in the medial axis M_k of finite set $K \subset \mathbb{R}^n$, there exists a positive δ.

Proposition 4.1.2. *Suppose $K \subset \mathbb{R}^n$ is finite set and the number of points in K is at least 2. For a given $x \in M_K$ there exists $\delta(x) > 0$ such that*

$$B\big(x, r(x) + \delta(x)\big) \cap K = \bar{B}\big(x, r(x)\big) \cap K = K(x). \tag{4.1.13}$$

Proof. Since K is finite, then $M_K \cap K = \emptyset$. Given that $x \in M_K$, we have $|x-y| > 0$ for each $y \in K$. Since $r(x) := \mathrm{dist}\big(x, K(x)\big)$,

$$r(x) = \min_{y \in K(x)} |x-y|.$$

First let us suppose that $K \neq K(x)$, then since K is finite,

$$r(x) < \min_{y \in K \setminus K(x)} |x-y|. \tag{4.1.14}$$

So if we define

$$\delta(x) := \min_{y \in K \setminus K(x)} |x-y| - r(x).$$

Then inequality (4.1.14) implies that $\delta(x) > 0$. Hence it then follows that if $K \neq K(x)$, then

$$\bar{B}\big(x, r(x) + \delta(x)\big) \cap \big(K \setminus K(x)\big) \neq \emptyset,$$

and

$$\bar{B}\big(x, r(x) + \delta(x)\big) \cap K = \bar{B}\big(x, r(x)\big) \cap K = K(x).$$

□

We show that in a small neighbourhood of a point x in the medial axis M_K, the distance of any point in the neighbhourhood to the finite subset $K(x)$ of the finite set K is the same as its distance to the set K.

Lemma 4.1.3. *Suppose K be a finite subset of $\mathbb{R}^n$, $x \in M_K$ and $\delta(x) > 0$ is given by the Proposition 4.1.2. Let $\epsilon(x) := \frac{\delta(x)}{8}$ and let $y \in \bar{B}\big(x, \epsilon(x)\big)$. If $K(x) \neq K$, then for all $z \in \bar{B}\big(y, \epsilon(x)\big)$*

$$dist\big(z, K(x)\big) < dist\big(z, K \setminus K(x)\big).$$

Proof. For simplicity, take $x = 0$ and suppose $K(x) \neq K$. Let $y \in \bar{B}\big(0, \epsilon(0)\big)$ and $z \in \bar{B}\big(y, \epsilon(0)\big)$. So

$$\begin{aligned} \mathrm{dist}\big(z, K(0)\big) &\leq |z| + \mathrm{dist}\big(0, K(0)\big) \\ &\leq |z - y| + |y| + \mathrm{dist}\big(0, K(0)\big) \\ &\leq 2\epsilon(0) + r(0). \end{aligned} \tag{4.1.15}$$

On the other hand, if $x^* \in K \setminus K(0)$, then it follows from Proposition 4.1.2 that

$$|x^*| = |x^* - 0| \geq r(0) + \delta(0),$$

so

$$|x^* - z| \geq |x^*| - |z| \geq r(0) + \delta(0) - 2\epsilon(0)$$

Therefore, since $\delta(0) = 8\epsilon(0)$,

$$|x^* - z| \geq r(0) + 6\epsilon(0) \tag{4.1.16}$$

Hence by the inequalities 4.1.15 and 4.1.16, we have

$$\text{dist}\big(z, K(0)\big) + 4\epsilon(0) \leq |x^* - z|,$$

which implies that

$$\text{dist}\big(z, K(0)\big) < \text{dist}\big(z, K \setminus K(0)\big),$$

for all $z \in \bar{B}\big(y, \epsilon(0)\big)$, from which the result follows. Consequently,

$$\text{dist}\big(z, K\big) = \text{dist}\big(z, K(x)\big),$$

for all $z \in \bar{B}\big(y, \epsilon(x)\big)$.

$\square$

The local representation theorem 4.1.4 shows that the lower transform $C_\lambda^l dist^2(\cdot, K)$ of the squared distance function to the finite set K for sufficiently large $\lambda > 0$ has the same representation as that of $C_\lambda^l dist^2(\cdot, K(x))$.

Theorem 4.1.4 (Local Representation Theorem). *Under the assumptions of Lemma 4.1.3; if $\sqrt{\lambda} > \max\{2, \frac{12r(x)}{\epsilon(x)}\}$, (since $\lambda > 4$ for Theorem 4.1.1), then*

$$C_\lambda^l dist^2\big(y, K\big) = C_\lambda^l dist^2\big(y, K(x)\big) = \frac{\lambda r^2}{1+\lambda} + (1+\lambda)\, dist^2\Big(y, \frac{C(K(x))}{1+\lambda}\Big) - \lambda(y-x)^2,$$

for all $y \in \bar{B}(x, \epsilon(x))$.

Proof. Let $z_1, ..., z_k \in \mathbb{R}^n$, $2 \leq k \leq n+1$, be such that

$$C_\lambda^l \text{dist}^2\big(y, K\big) = \sum_{i=1}^{k} \tau_i[\text{dist}^2\big(z_i, K\big) + \lambda |z_i - y|^2] \qquad (4.1.17)$$

for $\tau_1, ..., \tau_k > 0$, $\sum \tau_j = 1$ and $\sum \tau_j z_j = y$. Then by Theorem 4.1.1

$$\begin{aligned} |z_i - y| &\leq \frac{6}{\sqrt{\lambda}} \text{dist}\big(y, K\big) \\ &\leq \frac{6}{\sqrt{\lambda}} \Big(|y - x| + \text{dist}\big(x, K\big)\Big) \\ &\leq \frac{6}{\sqrt{\lambda}} \Big(\epsilon(x) + r(x)\Big) \end{aligned}$$

Since we know from Lemma 4.1.3 that $\epsilon(x) = \frac{\delta(x)}{8} < r(x)$, we have

$$\frac{6}{\sqrt{\lambda}}\Big(\epsilon(x) + r(x)\Big) \leq \frac{12}{\sqrt{\lambda}} r(x) < \epsilon(x)$$

provided $\sqrt{\lambda} > \frac{12r(x)}{\epsilon(x)}$. Furthermore,

$$\mathrm{dist}\big(z_i, K\big) = \mathrm{dist}\big(z_i, K(x)\big),$$

for $i = 1, ..., k$, and it follows that

$$C^l_\lambda dist^2\Big(y, K\Big) = C^l_\lambda dist^2\Big(y, K(x)\Big).$$

Hence from Lemma 3.1.2 of Chapter 3, it follows that

$$C^l_\lambda dist^2\big(y, K(x)\big) = \frac{\lambda}{1+\lambda} r^2 + (1+\lambda)\mathrm{dist}^2\Big(y, \frac{C(K(x))}{1+\lambda}\Big) - \lambda(y-x)^2$$

for all $y \in \bar{B}\big(x, \epsilon(x)\big)$. □

We need to provide some notation and results before we prove the semi-global property of the lower transform of squared distance function. Let Ω be a subset of $\mathbb{R}^2$, and for $\delta > 0$ and $R > 0$, define

$$\begin{aligned} (\Omega)_\delta &:= \big\{x \in \mathbb{R}^2, dist(x, \Omega) \leq \delta\big\} \\ M^*_{K,R} &:= \big\{x \in M_K \cap \bar{B}(0, R) : \#K(x) \geq 3\big\} \end{aligned}$$

We also define that for $\epsilon_1 > 0$ and $\epsilon_2 > 0$

$$W_{\epsilon_1, \epsilon_2} = \Big(\big(M^*_{K,R}\big)_{\epsilon_1} \cup \big(M_K \setminus \big(M^*_{K,R}\big)_{\epsilon_1}\big)_{\epsilon_2}\Big) \cap \bar{B}(0, R).$$

The following lemma will be helpful in the proof of the semi global result Theorem 4.1.6.

Lemma 4.1.5. *Suppose that $y \notin W_{\epsilon_1, \epsilon_2}$ and $y \in \bar{B}(0, R)$. Then $\delta > 0$ given by the Proposition 4.1.2 can be chosen independent of y.*

Proof. Suppose that y_n be a sequence such that $y_n \notin W_{\epsilon_1, \epsilon_2}$ and $y_n \in \bar{B}(0, R)$. Then we have corresponding sequence of points $x^n \in K$ such that

$$\mathrm{dist}^2\Big(y_n, K\Big) = |y_n - x_n|.$$

Suppose there exists a sequence $z_n \in K$ and $\delta_n \to 0$ as $n \to \infty$ such that

$$|y_n - z_n| = |y_n - x_n| + \delta_n$$

where $x_n \neq z_n$. The sequence y_n belong to a bounded region, that is, $y_n \in \bar{B}(0, R)$ and so for sufficiently large n the sequences converge, that is, $y_n \to y, x_n \to x, z_n \to z$. Then the limiting value gives

$$|y_n - z_n| = |y - z| = |y - x| = |y_n - x_n|$$

and $x \neq z$. This implies that y is equidistant from two points of K, hence $y \in M_K$ which is a contradiction to the assumption that $y \in \bar{B}(0, R)$ and $y \notin W_{\epsilon_1,\epsilon_2}$. Thus δ_n cannot tend to zero as $n \to \infty$ and therefore we can choose δ independent of y in $B(x, \delta)$. □

Theorem 4.1.6 (Semi-Global Theorem). *Suppose $K \subset \mathbb{R}^2$ is finite and $C(K) \subset \bar{B}(0, R)$ for a fixed radius $R > 0$. Then there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for sufficiently large $\lambda > 0$, $C^l_\lambda dist^2(y, K)$ can be calculated explicitly for either $y \in B(x, \epsilon_2)$ and $x \in M_K \setminus (M^*_{K,R})_{\epsilon_1}$, or $y \in B(x, \epsilon_1)$ and $x \in M^*_{K,R}$:*

$$C^l_\lambda dist^2(y, K) = \frac{\lambda}{1+\lambda} r^2 + (1+\lambda) dist^2\Big(y, \frac{C(K(x))}{1+\lambda}\Big) - \lambda(y - x)^2. \tag{4.1.18}$$

and if $y \notin W_{\epsilon_1,\epsilon_2}$, then

$$C^l_\lambda dist^2(y, K) = dist^2(y, K).$$

Proof. There are finitely many points $z_1, ..., z_j \in M^*_{K,R}$. Take, $\epsilon_1 = \min_{1 \le i \le k} \{\epsilon(z_i)\}$ and suppose $\epsilon(z_i) = \frac{\delta(z_i)}{8}$, where $\delta(z_i)$ is given by Proposition 4.1.2. Then it follows from Theorem 4.1.4 that if $\sqrt{\lambda} > \frac{2R}{\epsilon_1}$, that (4.1.18) holds when $x = z_i$ and $y \in B(z_i, \epsilon_1)$.

Suppose, for contradiction, that there exists a sequence $(z_k)_{k \in \mathbb{N}}$, $z_k \in M_K \setminus (M^*_{K,R})_{\epsilon_1}$ such that $\delta(z_k) \to 0$ as $k \to \infty$, where $\delta(z_k)$ is given by Proposition 4.1.2.

Now suppose two subsequences that for $y_j^{(1)} \neq y_j^{(2)}$, we have

$$|z_j - y_j^{(1)}| = r(z_j),\ |z_j - y_j^{(2)}| = r(z_j),\ |z_j - y_j^{(3)}| = r(z_j) + \delta(z_j),$$

where $y_j^{(1)}, y_j^{(2)} \in K(z_j)$, and $z_j \in M_K \cap \bar{B}(0, R) \setminus \bigcup_{i=1}^k B(z_i, \epsilon_1) \subset M_K \setminus M^*_{K,R}$ and $y_j^{(3)} \in K \setminus K(z_i)$. Since these sequences lie in a closed bounded domain, they have convergent subsequences, so we can re-label the sequences to ensure that $y_j^{(1)} \to y^{(1)}, y_j^{(2)} \to$

$y^{(2)}, y_j^{(3)} \to y^{(3)}$ and $z_j \to z_0$ as $j \to \infty$, for some $y^{(1)}, y^{(2)}, y^{(3)} \in K$ and $z_0 \in M_K \setminus M^*_{K,R}$. Furthermore, since K is a finite set, it follows that for j sufficiently large,

$$y^{(1)} = y_j^{(1)} \neq y_j^{(2)} = y^{(2)}$$
$$y^{(3)} = y_j^{(3)} \neq y_j^{(1)} = y^{(1)}$$
$$y^{(3)} = y_j^{(3)} \neq y_j^{(2)} = y^{(2)}$$

and so

$$|z_j - y^{(3)}| = |z_j - y_j^{(3)}| = r(z_j) + \delta(z_j) = dist(z_j, K) + \delta(z_j) \tag{4.1.19}$$

whereas

$$\begin{aligned} dist(z_j, K) &= |z_j - y^{(1)}| = |z_j - y^{(2)}| \\ dist(z_0, K) &= |z_0 - y^{(1)}| = |z_0 - y^{(2)}|. \end{aligned}$$

Therefore, as $j \to \infty$, then 4.1.19 implies that

$$|z_0 - y^{(3)}| = r(z_j) = \mathrm{dist}(z_0, K) = |z_0 - y^{(1)}| = |z_0 - y^{(2)}|.$$

Thus z_0 is reached by three points in K, which is a contradiction because $z_0 \in M_K \setminus M^*_{K,R}$. So we define

$$\delta_2 := \inf\left\{\delta(z) : \ z \in M_K \setminus \left(M^*_{K,R}\right)_{\epsilon_1} \cap \bar{B}(0, R)\right\}.$$

Let us suppose that $\epsilon_2 = \frac{\delta_2}{8}$. Then using the local representation Theorem 4.1.4, it follows that if $\sqrt{\lambda} > \frac{2R}{\epsilon_2}$, then 4.1.18 holds when $x \in M_K \setminus \left(M^*_{K,R}\right)_{\epsilon_1} \cap \bar{B}(0, R)$ and $y \in B(x, \epsilon_2)$.

We now prove that if $y \notin W_{\epsilon_1, \epsilon_2}$, then

$$C^l_\lambda \mathrm{dist}^2(y, K) = \mathrm{dist}^2(y, K), \qquad \text{for } \lambda > 0 \text{ sufficiently large.}$$

Let us suppose $y \notin W_{\epsilon_1, \epsilon_2}$. Then $dist(y, K) = |y - x_k|$ for some fixed $x_k \in K$ and for $x^* \in K \setminus \{x_k\}$,

$$|y - x^*| \geq |y - x_k| + \delta$$

for some $\delta > 0$ which can be chosen independently using argument of Lemma 4.1.5. Therefore, if $y \in B(x, \epsilon)$ where $\epsilon = \frac{\delta}{8}$, then

$$|y - x_k| = \text{dist}(y, x_k) \leq |y - x| + |x - x_k| < \epsilon + |x - x_k|$$

whereas if $x^* \in K \setminus \{x_k\}$, then

$$|y - x^*| \geq |x - x^*| - |y - x| \geq |x - x_k| + \delta - \epsilon$$

Therefore,

$$|y - x_k| < |y - x^*|$$

provided

$$\epsilon + |x - x_k| \leq |x - x_k| + \delta - \epsilon \iff \epsilon \leq \frac{\delta}{2}.$$

Since $\epsilon = \frac{\delta}{8}$, therefore, if $z \in B(y, \epsilon)$, then

$$\text{dist}(z, K) = |z - x_k|$$

Thus,

$$\text{dist}^2(z, K) = |z - x_k|^2, \; z \in B(y, \epsilon).$$

This implies that $\text{dist}^2(\cdot, K) \in C^\infty(\mathbb{B}(y, \epsilon))$. Thus from Zhang [20, Theorem 2.3(iv)], we have that

$$C_\lambda^l \text{dist}^2(y, K) = \text{dist}^2(y, K) \quad \text{whenever} \quad \lambda > \max\left\{C(y), \frac{1}{\delta^2}\left(1 + \left|D\text{dist}^2(y, K)\right|\right)\right\},$$

where $C(y)$ is Lipschitz constant. We now estimate the value of $\lambda > 0$ here. Consider the function

$$f(z) := \text{dist}^2(z, K) = |z|^2 - 2zx_k + |x_k|^2,$$

and $\frac{\partial f}{\partial z_j} = 2z_j - 2(x_k)_j$ and $Df(z) = 2(z - x_k)$. Therefore,

$$|Df(z) - Df(y)| \leq C(y)|z - y|,$$

for all $z \in B(y, \epsilon)$ and $C(y) = 2$ is Lipschitz constant as follows

$$Df(z) - Df(y) = 2(z - x_k) - 2(y - x_k) = 2(z - y).$$

The modulus of the gradient of the squared distance function is bounded because $|y|$ is bounded by R if $y \in \bar{B}(0,R)$. Hence $\max\left\{2, \frac{1}{\delta^2}\left(1 + \left|D\mathrm{dist}^2(y,K)\right|\right)\right\}$ is bounded above independently of $y \notin W_{\epsilon_1,\epsilon_2}$ and $y \in \bar{B}(0,R)$. Thus λ can be chosen large enough to ensure that

$$C^l_\lambda \mathrm{dist}^2\big(y,K\big) = \mathrm{dist}^2\big(y,K\big).$$

for all $y \notin W_{\epsilon_1,\epsilon_2}$. □

We now establish a global result for triangulation simplices based on the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to a finite set $K \subset \mathbb{R}^n$. This theorem is motivated by the fact that we are interested in surface reconstruction of a surface fitting a large finite number of points. We show that for a single simplex $C(K_s)$ for $K_s \subset K$ in a collection of trinagulated simplices of $C(K)$, the lower transform of the squared distance function to K equal the lower transform to set K_s and thus it holds for all the simplices in the triangulation. In future, we will show explore that if K has a regular Delaunay triangulation, then Theorem 4.1.7 will hold.

Theorem 4.1.7. *Suppose $K \subset \mathbb{R}^n$ is finite and let $\dim C(K) = m$, and $C(K)$ be trianglualted into a collection S of simplices, that is, $C(K) = \underset{s}{\cup}\Delta_s$.*

(i) If for a simplex $\Delta_s = C(K_s)$ in collection S of simplices with $0 < \dim \Delta_s \leq m$, where $K_s = \big\{x_1, \ldots, x_{s+1}\big\} \subset K$,

$$dist(x,K) = dist(x,K_s), \tag{4.1.20}$$

for x in a neighbourhood of Δ_s^δ (for $\delta > 0$, $\Delta_s^\delta = \{x \in \mathbb{R}^n, dist(x,\Delta_s) \leq \delta\}$), then for $\lambda > 0$ sufficiently large,

$$C^l_\lambda dist^2(x,K) = C^l_\lambda dist^2(x,K_s).$$

(ii) If $dist(x,K) = dist(x,K_s)$ for x in a neighbourhood of Δ_s for all simplices $\Delta_s \in S$, which are contained in the triangulation S, then

$$C^l_\lambda dist^2(x,K) = C^l_\lambda dist^2(x,K_s),$$

for $x \in \Delta_s$ when $\lambda > 0$ is sufficiently large for every $\Delta_s \in S$.

Proof of (i). Since $\text{dist}(x,K) = \text{dist}(x,K_s)$ if $\text{dist}(x,\Delta_s) \le \delta$, then for any $\hat{x} \in \Delta_s$, by Theorem 4.1.1, $\exists\, x_1, \ldots, x_k \in \mathbb{R}^n$, $1 \le k \le n+1$, $\tau_1, ..., \tau_k > 0, \sum_{i=1}^k \tau_i = 1, \sum_{i=1}^k \tau_i x_i = \hat{x}$ such that

$$C_\lambda^l \text{dist}^2(x,K) = \sum_{i=1}^k \tau_i \big(\text{dist}^2(x_i,K) + \lambda|x_i - \hat{x}|^2.$$

Furthermore, we have

$$|x_i - \hat{x}| \le \frac{6}{\sqrt{\lambda}} \text{dist}(\hat{x},K), \forall\, i = 1,2,...,k.$$

Since $\hat{x} \in \Delta_s$, let $M_s = \sup_{x\in\Delta_s}\{\text{dist}(x,K)\} < +\infty$, so that $|x_i - \hat{x}| \le \frac{6}{\sqrt{\lambda}} M_s$. Hence when $\lambda > \frac{(6M_s)^2}{\delta^2}$, $|x_i - \hat{x}| < \delta$, $i = 1,2,...,k$. So by our assumption (4.1.20),

$$\begin{aligned} C_\lambda^l \text{dist}^2(x,K) &= \sum_{i=1}^k \tau_i \big(\text{dist}^2(x_i,K) + \lambda|x_i - \hat{x}|^2\big) \\ &= \sum_{i=1}^k \tau_i \big(\text{dist}^2(x_i,K_s) + \lambda|x_i - \hat{x}|^2\big) \\ &\ge C\big(\text{dist}^2(x_i,K_s) + \lambda|x_i - \hat{x}|^2\big)\big|_{x=\hat{x}} \\ &= C_\lambda^l \text{dist}^2(\hat{x},K_s). \end{aligned}$$

On the other hand, obviously, $C_\lambda^l \text{dist}^2(\hat{x},K) \le C_\lambda^l \text{dist}^2(\hat{x},K_s)$ as $K_s \subset K$. Thus

$$C_\lambda^l \text{dist}^2(\hat{x},K) = C_\lambda^l \text{dist}^2(\hat{x},K_s), \qquad \text{for all } \hat{x} \in \Delta_s.$$

Proof of (ii): We notice for each $\Delta_s = C(K_s) \in S$, there is $\delta_s > 0$ such that

$$\text{dist}^2(x,K) = \text{dist}^2(x,K_s), \text{ if } \text{dist}(x,\Delta_s) \le \delta_s. \tag{4.1.21}$$

Since the collection of simplices is finite, let $\delta_0 = \min_s \delta_s > 0$, for $x \in \Delta_s^{\delta_0}$, (4.1.21) holds for all $\Delta_s \in S$ with δ_s replaced by $\delta_0 > 0$. The proof then follows from that of (i). □

Example 4.1.8. Consider set $K = \{(-1,0),(0,1),(0,-1),(1,0)\} \subset \mathbb{R}^2$. We plot $C(K)$ and triangulate $C(K)$ into a collection S of simplices $C(K) = \Delta_{s_i}^1 \cup \Delta_{s_i}^2$ for $i = 1, ..., 8$ in

Figure 4.1. The collection S of simplices $\Delta^1_{s_i}$ and $\Delta^2_{s_i}$ for $i = 1, ..., 8$ where

$$\begin{aligned}
\Delta^1_{s_1} &= C(K_{s_1}) \text{ where } K_{s_1} = \{(1,0),(0,1)\}\\
\Delta^1_{s_2} &= C(K_{s_2}) \text{ where } K_{s_2} = \{(1,0),(0,-1)\}\\
\Delta^1_{s_3} &= C(K_{s_3}) \text{ where } K_{s_3} = \{(-1,0),(0,1)\}\\
\Delta^1_{s_4} &= C(K_{s_4}) \text{ where } K_{s_4} = \{(-1,0),(0,-1)\}\\
\Delta^2_{s_1} &= C(K_{s_5}) \text{ where } K_{s_5} = \{(-1,0),(0,1),(1,0)\}\\
\Delta^2_{s_2} &= C(K_{s_6}) \text{ where } K_{s_6} = \{(0,1),(0,-1),(1,0)\}\\
\Delta^2_{s_3} &= C(K_{s_7}) \text{ where } K_{s_7} = \{(-1,0),(0,-1),(1,0)\}\\
\Delta^2_{s_4} &= C(K_{s_8}) \text{ where } K_{s_8} = \{(-1,0),(0,1),(0,-1)\}
\end{aligned}$$

Obviously we note that for x in a neighbourhood of $\Delta^1_{s_i}$ or $\Delta^2_{s_i}$,

$$\mathrm{dist}(x, K) = \mathrm{dist}(x, K_{s_i}), \quad \text{for} i = 1, ..., 8.$$

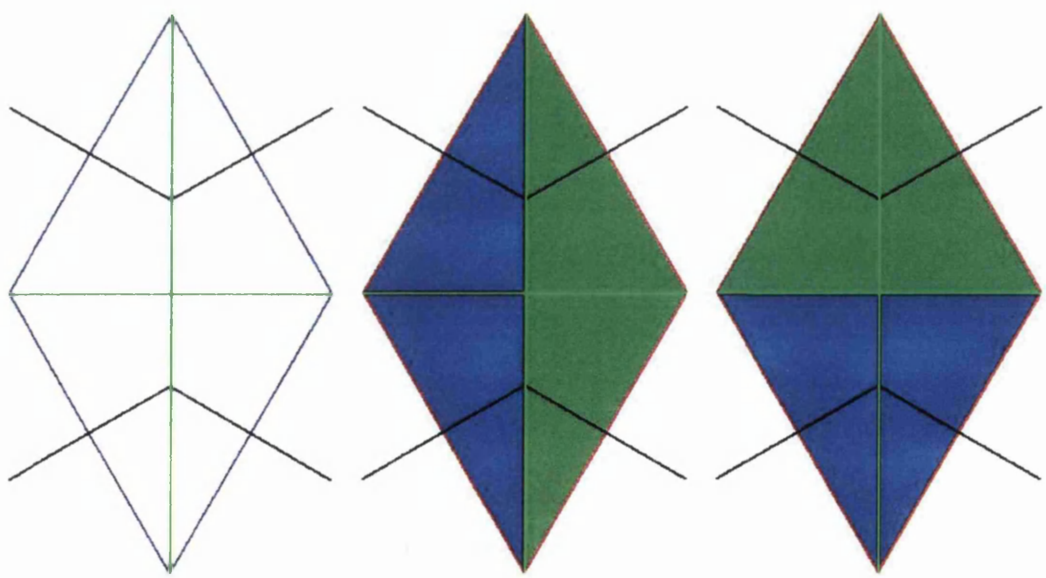

Figure 4.1: Triangulation of $C(K) = \Delta^1_{s_i} \cup \Delta^2_{s_i}$ for $i = 1, ..., 8$ into simplices.

Remark 4.1.9. This result will be used in Chapter 5 for surface reconstruction. We will explore, when the regularity condition holds for triangulations in the future.

We mainly have studied the local properties of the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the squared distance function to a finite $K \subset \mathbb{R}$. The semi-global property of lower transform of the squared distance function to a finite set $K \subset \mathbb{R}^2$ provides detailed information of how the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ modifies the squared distance functions and gives tight approximation property of the lower transform of the squared distance functions. We have established a global triangulation simplices property of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ to finite set $K \subset \mathbb{R}^n$. The surface reconstruction without all these properties would not have made sense, and thus are important properties.

4.2 Examples of explicit formulae of $C^l_\lambda \mathrm{dist}^2(\cdot, K)$

In this section we calculate explicit formulae for the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance functions to a finite subset K of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$. These concrete examples of the lower transforms of the squared distance functions to the finite sets will be useful in Chapter 5 and Chapter 6. We also plot some of the squared distance functions to the finite sets of these examples and their lower transforms. The purpose of these plots is to see the effect of smoothing of the lower transform to the original squared distance function. Plotting these graphs, we use both Mathematica and MATLAB programming code. In addition, we notice that the critical points for both the original squared distance functions and their lower transform are the same.

The first plot is of the squared distance function to the finite set $K = \{-1, 1\}$ and it's lower transform is Examlpe (3) taken from [21]. The squared distance function is $f(x) := \mathrm{dist}^2\big(x, K\big) = \min\big\{(x+1)^2, (x-1)^2\big\}$, and for $\lambda > 0$, the lower transform $h(x) := C^l_\lambda \mathrm{dist}^2\big(x, K\big)$ is given by

$$h(x) = \begin{cases} (x-1)^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda |x|^2 & |x| \le \frac{1}{1+\lambda} \\ (x+1)^2 & x < -\frac{1}{1+\lambda}. \end{cases} \tag{4.2.22}$$

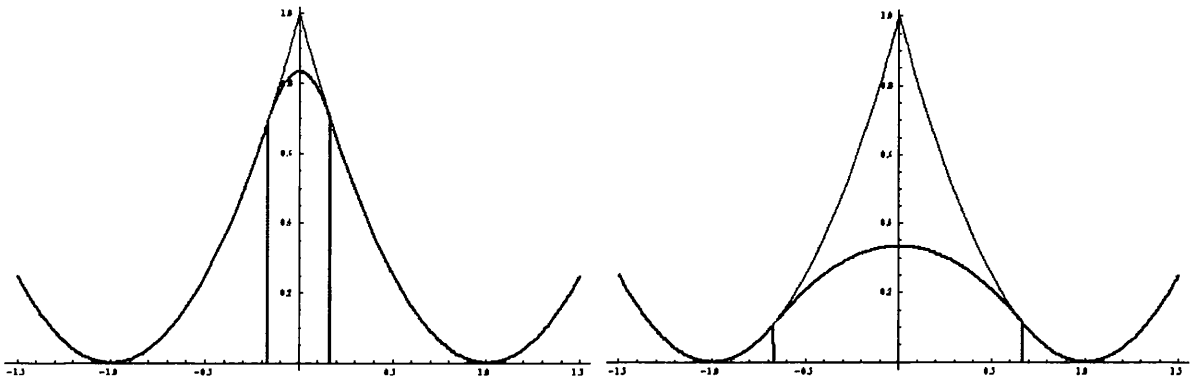

Figure 4.2: From left to right; plots of the squared distance function $f(x)$ and its lower transform $h(x)$ for $\lambda = 5$ and $\lambda = 0.5$.

In Figure 4.2, we plot the graph of the squared distance function $f(x)$ and its lower transform $h(x)$ for $\lambda = 5$ and $\lambda = 0.5$ in the domain $|x| \leq 1.5$. We note from the graph that the lower transform has simply smoothed the corner (i.e., near the singulatry) and made the resulting function $C^{1,1}(\mathbb{R})$-smooth. Note that, the modified (i.e., smoothed) region reduces with the increase in the value of $\lambda > 0$ and hence resulting a tight smoothing. It can be seen clearly that $h(x) \to f(x)$ uniformly as $\lambda \to +\infty$ and, in fact, that $h'(x) \to f'(x)$ except at 0. This example has great importance in applications of the lower transforms of squared distance function to finite sets in Chapter 5 and Chapter 6.

Example 4.2.1. In this example we consider the squared distance function $f(x, y)$ to the finite set $K = \{(0, -1), (0, 1)\}$, which is defined by

$$f(x, y) := \min\left\{x^2 + (y+1)^2, x^2 + (y-1)^2\right\},$$

and the explicit formula of its lower transform $h(x, y) := C^l_\lambda \mathrm{dist}^2\big((x, y), K\big)$ for $\lambda > 0$ is given by

$$h(x, y) = \begin{cases} x^2 + (y-1)^2 & y > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + x^2 - \lambda|y|^2 & |y| \leq \frac{1}{1+\lambda} \\ x^2 + (y+1)^2 & y < -\frac{1}{1+\lambda} \end{cases} \tag{4.2.23}$$

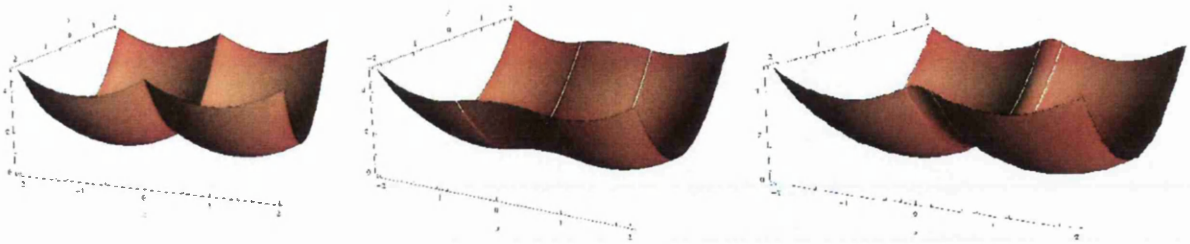

Figure 4.3: From left to right; plot of the squared distance function $f(x,y)$ and its lower transform $h(x,y)$ for $\lambda = 0.5$ and $h(x,y)$ for $\lambda = 5$.

In Figure 4.3, we plot the graph of the squared distance function $f(x,y)$ and its lower transform $h(x,y)$ for $\lambda = 5$ and $\lambda = 0.5$ in the domain $|y| \leq 2$ and $|x| \leq 2$. We note from the graph that the lower transform has simply smoothed the small corner that is, near the singulatry $(0,0)$). Note that, the modified (i.e., smoothed) region reduces with the increase in the value of $\lambda > 0$ and hence resulting a tight smoothing.

Example 4.2.2. In this 2D example, let us consider the squared distance function $f(x,y)$ to the finite set $K = \big\{(-1,0),(1,0),(0,1)\big\}$, which is defined by

$$f(x,y) := \min\big\{(x+1)^2 + y^2, (x-1)^2 + y^2, x^2 + (y-1)^2\big\},$$

and the explicit formula of its lower transform $h(x,y) := C^l_\lambda\big(f(x,y)\big)$ for $\lambda > 0$, is then given by

$$h(x,y) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2+y^2) & |x| \geq y - \frac{1}{1+\lambda}, y \geq 0 \\ x^2 + (y-1)^2 & |x| \leq y - \frac{1}{1+\lambda}, y \geq \frac{1}{1+\lambda} \\ (x+1)^2 + y^2 & x \leq -\frac{1}{1+\lambda}, y \leq -x - \frac{1}{1+\lambda} \\ (x-1)^2 + y^2 & x \geq \frac{1}{1+\lambda}, y \leq x - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{2}\big(x + y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2) & |y - x| < \frac{1}{1+\lambda}, y > -x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{2}\big(-x + y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2) & |x + y| < \frac{1}{1+\lambda}, y > x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + y^2 - \lambda x^2 & |x| < \frac{1}{1+\lambda}, y < 0 \end{cases} \tag{4.2.24}$$

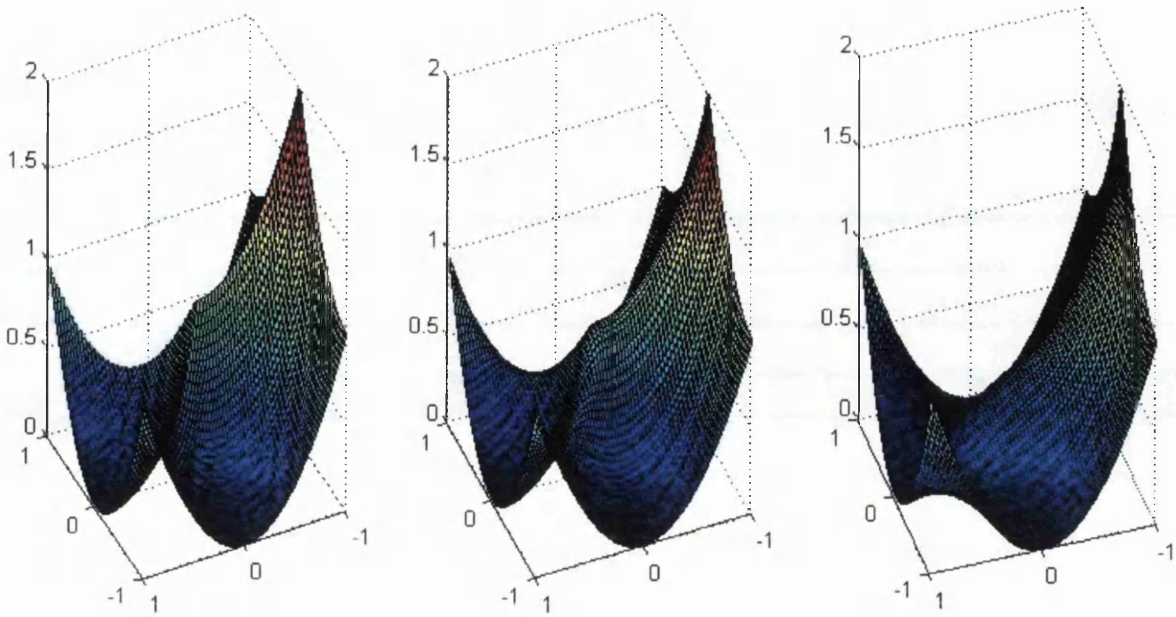

Figure 4.4: From left to right; plots of the squared distance function $f(x, y)$ and its lower transform $h(x, y)$ for $\lambda = 5$ and $\lambda = 0.5$.

Example 4.2.3. Let the squared distance function to $K = \{(0,1), (\frac{\sqrt{3}}{2}, -\frac{1}{2}), (-\frac{\sqrt{3}}{2}, -\frac{1}{2})\}$ defined by $f(x, y) := \min\{x^2 + (y-1)^2, (x - \frac{\sqrt{3}}{2})^2 + (y + \frac{1}{2})^2, (x + \frac{\sqrt{3}}{2})^2 + (y + \frac{1}{2})^2\}$, and its lower transform defined by $h(x, y) := C^l_\lambda(f(x, y))$. Then for $\lambda > 0$ we have

$$h(x,y) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2 + y^2) & |x| \geq \frac{1}{\sqrt{3}}(y - \frac{1}{1+\lambda}), y \geq \frac{-1}{2(1+\lambda)} \\ x^2 + (y-1)^2 & |x| \leq \sqrt{3}(y - \frac{1}{1+\lambda}), y > \frac{1}{1+\lambda} \\ (x + \frac{\sqrt{3}}{2})^2 + (y + \frac{1}{2})^2 & x < \frac{-\sqrt{3}}{2(1+\lambda)}, y \leq \frac{-1}{\sqrt{3}}x - \frac{1}{1+\lambda} \\ (x - \frac{\sqrt{3}}{2})^2 + (y + \frac{1}{2})^2 & x > \frac{\sqrt{3}}{2(1+\lambda)}, y \leq \frac{1}{\sqrt{3}}x - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{4}(\sqrt{3}x + y - \frac{1}{1+\lambda})^2 - \lambda(x^2 + y^2) & \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)} < x < \frac{\sqrt{3}(1+y(1+\lambda))}{1+\lambda}, \\ & y < \frac{1}{\sqrt{3}}x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{4}(-\sqrt{3}x + y - \frac{1}{1+\lambda})^2 - \lambda(x^2 + y^2) & \frac{\sqrt{3}(1+y(1+\lambda))}{1+\lambda} < x < \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)}, \\ & y < -\frac{1}{\sqrt{3}}x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + (1+\lambda)(y + \frac{1}{2(1+\lambda)})^2 - \lambda(x^2 + y^2) & |x| < \frac{\sqrt{3}}{2(1+\lambda)}, y < -\frac{1}{2(1+\lambda)} \end{cases} \tag{4.2.25}$$

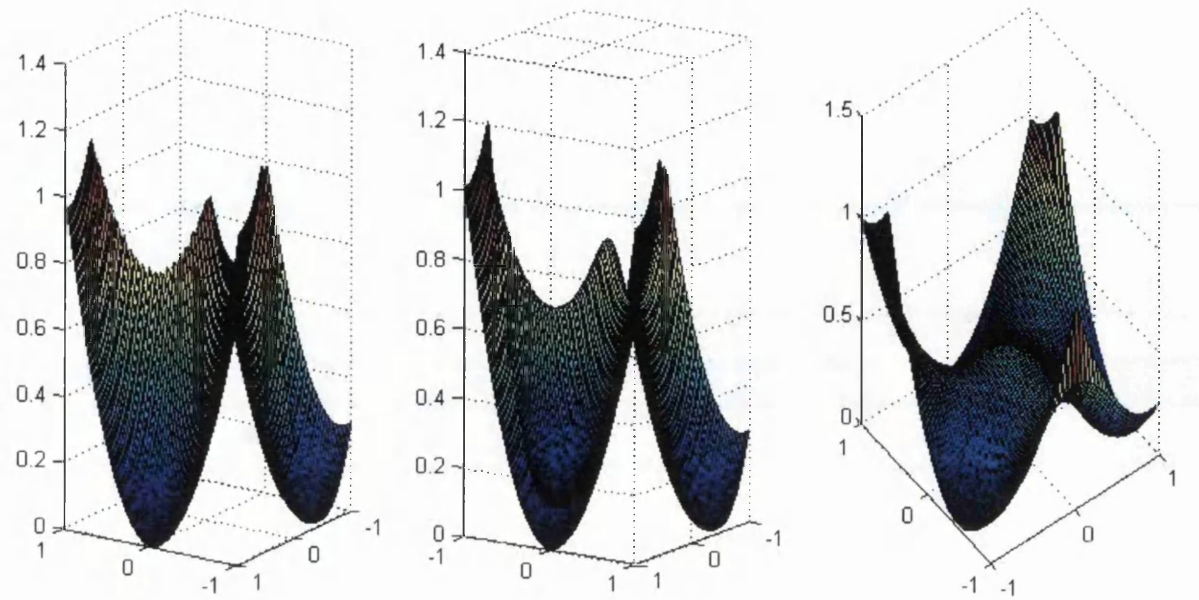

Figure 4.5: The MATLAB programming plots from left to right of the squared distance function $f(x,y)$ to the finite set K and its lower transform $h(x,y)$ for $\lambda = 5$ and $\lambda = 0.5$.

Example 4.2.4. Let us consider the squared distance function $f(x,y,z)$ to set $K = \big\{(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1),(-1,1,1),(-1,1,-1),(-1,-1,1),(-1,-1,-1)\big\}$.Then we calculate the lower transform $h(x,y,z) := C^l_\lambda\big(f(x,y,z)\big)$ for $\lambda > 0$,

$$h = \begin{cases} \frac{3\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \big(|x|-1\big)^2 + \big(|y|-1\big)^2 + \big(|z|-1\big)^2 & |x| > \frac{1}{1+\lambda}, |y| > \frac{1}{1+\lambda}, |z| > \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\big(|y| - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| > \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\big(|x| - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & |x| > \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\big(|z| - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| > \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\Big(\big(|x| - \frac{1}{1+\lambda}\big)^2 + \big(|y| - \frac{1}{1+\lambda}\big)^2\Big) - \lambda(x^2+y^2+z^2) & |x| \ge \frac{1}{1+\lambda}, |y| \ge \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\Big(\big(|x| - \frac{1}{1+\lambda}\big)^2 + \big(|z| - \frac{1}{1+\lambda}\big)^2\Big) - \lambda(x^2+y^2+z^2) & |x| \ge \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \ge \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\Big(\big(|y| - \frac{1}{1+\lambda}\big)^2 + \big(|z| - \frac{1}{1+\lambda}\big)^2\Big) - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \ge \frac{1}{1+\lambda}, |z| \ge \frac{1}{1+\lambda} \end{cases} \tag{4.2.26}$$

Example 4.2.5. Let set $K = \Big\{\Big(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\Big), \Big(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\Big), (0,0,1), (0,0,-1)\Big\}$ and the squared distance function to K is defined by

$$\begin{aligned} f(x,y,z) \ &:= \ \min\Big\{\Big(x-\frac{\sqrt{3}}{2}\Big)^2+\Big(y-\frac{1}{2}\Big)^2+z^2, \Big(x-\frac{\sqrt{3}}{2}\Big)^2+\Big(y+\frac{1}{2}\Big)^2+z^2, \\ & \qquad x^2+y^2+(z-1)^2, x^2+y^2+(z+1)^2\Big\}. \end{aligned}$$

Then, we calculate the lower transform $h(x,y,z) := C^l_\lambda\big(f(x,y,z)\big)$ for $\lambda > 0$,

$$h = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2) & (1) \\ f(x,y,z) & (2) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}(-x+\sqrt{3}y)^2 - \lambda(x^2+y^2+z^2) & (3) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}(x+\sqrt{3}y)^2 - \lambda(x^2+y^2+z^2) & (4) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{7}(2x+\sqrt{3}z - \frac{\sqrt{3}}{(1+\lambda)})^2 - \lambda(x^2+y^2+z^2) & (5) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{7}(-2x+\sqrt{3}z + \frac{\sqrt{3}}{(1+\lambda)})^2 - \lambda(x^2+y^2+z^2) & (6) \\ \frac{\lambda}{1+\lambda} + (1+\lambda)(x^2+y^2) - \lambda(x^2+y^2+z^2) & (7) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{8}\Big((z-2y-\frac{1}{1+\lambda})^2 + (2x+\sqrt{3}z-\frac{\sqrt{3}}{1+\lambda})^2 + (x-\sqrt{3}y)^2\Big) - \lambda(x^2+y^2+z^2) & (8) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{8}\Big((z+2y-\frac{1}{1+\lambda})^2 + (2x+\sqrt{3}z+\frac{\sqrt{3}}{1+\lambda})^2 + (x-\sqrt{3}y)^2\Big) - \lambda(x^2+y^2+z^2) & (9) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}\Big((2x-\frac{\sqrt{3}}{1+\lambda})^2 + 4z^2\Big) - \lambda(x^2+y^2+z^2) & (10) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{8}\Big(\big(z-2y+\frac{1}{1+\lambda}\big)^2 + \big(\sqrt{3}z-2x+\frac{\sqrt{3}}{1+\lambda}\big)^2 + (x-\sqrt{3}y)^2\Big) - \lambda(x^2+y^2+z^2) & (11) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{8}\Big(\big(z+2y+\frac{1}{1+\lambda}\big)^2 + \big(\sqrt{3}z-2x+\frac{\sqrt{3}}{1+\lambda}\big)^2 + (x+\sqrt{3}y)^2\Big) - \lambda(x^2+y^2+z^2) & (12) \end{cases} \tag{4.2.27}$$

The formulae of the regions from (1) to (12) has not been included in order to provide a justified explicit formula of the lower transform $C^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big)$ of the squared distance function to the set K.

Example 4.2.6. Let set $K = \big\{\big(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big), \big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big), (0,1,0), (0,0,1)\big\}$ and the squared distance function is defined by

$$\begin{aligned} f(x,y,z) \ &:= \ \min\Big\{\Big(x-\frac{\sqrt{3}}{2}\Big)^2+\Big(y+\frac{1}{2}\Big)^2+z^2, \Big(x+\frac{\sqrt{3}}{2}\Big)^2+\Big(y+\frac{1}{2}\Big)^2+z^2, \\ & \qquad x^2+(y-1)^2+z^2, x^2+y^2+(z-1)^2\Big\}. \end{aligned}$$

Then, we calculate the lower transform $h(x,y,z) := C^l_\lambda\Big(f(x,y,z)\Big)$ for $\lambda > 0$,

$$h(x,y,z) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2) & (1) \\ (x-\frac{\sqrt{3}}{2})^2 + (y+\frac{1}{2})^2 + z^2 & (2) \\ (x+\frac{\sqrt{3}}{2})^2 + (y+\frac{1}{2})^2 + z^2 & (3) \\ x^2 + (y-1)^2 + z^2 & (4) \\ x^2 + y^2 + (z-1)^2 & (5) \\ \frac{\lambda}{1+\lambda} + (1+\lambda)z^2 - \lambda(x^2+y^2+z^2) & (6) \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{5}\big(\sqrt{3}x - y - z + \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & (7) \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{5}\big(\sqrt{3}x + y + z - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & (8) \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{5}\big(2y - z + \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & (9) \\ z^2 + \frac{1+\lambda}{4}\big(y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & (10) \end{cases} \tag{4.2.28}$$

Some of the explicit formulae of these examples of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to finite subset K of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, will be used repeatedly in applications of the lower transform and some we will consider for future research.

Chapter 5

Applications to curve and surface reconstruction in computer-aided design

In this chapter we discuss the applications of the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the squared distance functions to a finite set $K \subset \mathbb{R}^3$ to computer-aided design (CAD), and in particular, applications to surface reconstruction [6] (also see [7, 8, 12]). We modify an approach by T.K. Dey [6] using the squared distance function to finite sets in $\mathbb{R}^3$ by constructing the surface using instead a smooth approximation of the squared distance function to finite sets. The smooth approximation that we use is the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the squared distance function to the finite set K; the key tight approximation property of the lower transform is proved in [20, Theorem 2.3(iv)]; see also Chapter 2, Theorem 2.2.15.

5.1 Advantages of Using $C^l_\lambda \mathbf{dist}^2(\cdot, K)$ for Surface Reconstruction

In this section, the surface reconstruction using the squared distance $\mathrm{dist}^2(\cdot, K)$ is compared with the surface reconstruction using the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the squared distance function to the finite set K. The motivation comes from the fact that Dey's [6] approach uses the squared distance $\mathrm{dist}(\cdot, K)$ of a point to the closest point in a finite set $K \subset \mathbb{R}^3$, and so the gradient of the squared distance function is not well-defined at every point. In contrast to $\mathrm{dist}^2(\cdot, K)$, the lower transform $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the squared distance function to the finite set K is $C^{1,1}$ (see [1, Theorem 3.1]). Note that if f is a continously differentiable function ($f \in C^1$), then the direction of the steepest ascent of f is in the direction of the gradient of f. Thus the direction of steepest ascent of $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ is in the direction of the gradient of $C^l_\lambda \mathrm{dist}^2(\cdot, K)$. Hence by using theory of differential equations we easily get the direction of flow by calculating the gradient of $C^l_\lambda \mathrm{dist}^2(\cdot, K)$ of the squared distance function to the finite set K and associated manifolds. On the other hand, Dey has to work to define the flow and associated manifolds using the steepest ascent of the distance function, which are described in the following parapragh.

For every regular point in $\mathbb{R}^3$ there is a unique direction of steepest ascent of the distance function, and for the direction of steepest ascent and its critical points we include some preliminaries from [6]. For every point $x \in \mathbb{R}^3$, let a set of points $K(x) \subset K$ (see Definition 2.1.4) be such that it contains all the points of K with minimum distance to x. Recall that a point x is a geometric critical point if $x \in C\big(K(x)\big)$ and is said to be a regular point otherwise, where $C\big(K(x)\big)$ is the convex hull of K. For any point $x \in \mathbb{R}^3$, let $q(x) \in C\big(K(x)\big)$ be the closest point to x, then point $q(x)$ is called the driver of x (see Definition 2.3.5).

In [6, Lemma 10.1], it is shown that for any regular point $x \in \mathbb{R}^3$ the steepest ascent of the distance function at the point x is in the direction of $x - q(x)$.

A vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ is defined such that for all regular points $v(x) = \frac{x-q(x)}{\|x-q(x)\|}$ if $x \neq q(x)$ and 0 at the critical points.

A flow function $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ induced by the vector field v and determines how the flow ϕ varies with $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$ and for any critical point x

$$\phi(t, x) = x, \quad \forall\, t \in \mathbb{R}.$$

All this helps to define a flow curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ starting and ending at geomteric critical points. The reason Dey needs to define all the concepts above is because the squared distance function has points where the gradient does not exist.

On the other hand, if we consider the smoothed squared distance function to the finite set K and a point $x \in \mathbb{R}^3 \setminus M_K$ where M_K is the medial axis of K, then for $\lambda > 0$ sufficiently large,

$$C^l_\lambda \mathrm{dist}^2(x, K) = \mathrm{dist}^2(x, K)$$

proved in Chapter 4, Theorem 4.1.6. We note that for such x, the set $K(x)$ is a singleton set and the point $p \in K(x)$ is the driver of x in this case, that is, $p = q(x)$. Therefore, in such case the gradient at x is given by

$$DC^l_\lambda \mathrm{dist}^2(x, K) = D\mathrm{dist}^2(x, K) = 2(x - p).$$

Thus the gradient of $C^l_\lambda \mathrm{dist}^2(x, K)$ for a point away from medial axis will be pointing in the same direction in which $x - q(x)$ will be pointing.

We look deeper on the effects of how our particular smoothed squared distance function has helped to enhance the understanding of the dynamics of the models. The first order gradient system (5.1.1) of our particular smoothed squared distance function to a finite set K,

$$\begin{aligned} \dot{x}(t) &= DC^l_\lambda \mathrm{dist}^2(x(t), K) \\ x(0) &= u_0 \end{aligned} \tag{5.1.1}$$

has unique global solution. It is not true that a global solution for the non-smooth distance function always exists because the trajectory of $\dot{x} = D\mathrm{dist}(\cdot, K)$ will always remain inside a region where $\mathrm{dist}(\cdot, K)$ is smooth, bounded by parts of the medial axis. The reason for that is, for initial data away from medial axis, if the problem $\dot{x} = D\mathrm{dist}(\cdot, K)$ is locally well-posed (i.e., there exists a unique solution for small times), then for small times, Dey's method reduces to solving $\dot{x} = D\mathrm{dist}(\cdot, K)$ for small time. Dey's flow $\dot{x} = D\mathrm{dist}(\cdot, K)$ away from medial axis is ok but when the flow hits the medial axis, Dey's method required human interventions whereas our flow $\dot{x} = DC^l_\lambda \mathrm{dist}^2(\cdot, K)$ has the advantage that it does not need human interventions as well as it is globally well-posed. To show that the global solution for the non-smooth squared distance function does not exist, we provide the following simple example of two point set in $\mathbb{R}^2$.

Example 5.1.1. Let $K = \{(-1,0),(1,0)\}$ and let the initial value be $p := (\frac{1}{2}, \frac{1}{2})$. Then $D\mathrm{dist}^2(p, K) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ which is a unit vector. The flow will be $\dot{x} = D\mathrm{dist}^2(x, K) = D\mathrm{dist}^2(p, K) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ before it hit the medial axis. At the medial axis point $(1, 0)$, the new instruction is to change the direction to $(1, 0)$ and move upwards. When the flow hits the medial axis, Dey's approach calculates $C\big(K(x)\big)$ and find a point $y \in C(K(x))$ such that $\mathrm{dist}\big(x, C(K(x))\big) = |x - y|$, and then by using this point y to give new direction say $\dot{x} = \frac{y-x}{|y-x|}$.

On the other hand, using our particular smoothed squared distance function, we have $\dot{x} = \frac{1}{2} DC^l_\lambda \mathrm{dist}^2(x, K)$. We assume that $\frac{1}{2} > \frac{1}{1+\lambda}$ and $x = (x_1, x_2) \in \mathbb{R}^2$. The system is decoupled as

$$\begin{aligned} \dot{x}_1 &= x_1 - 1 \\ \dot{x}_2 &= x_2 \end{aligned}$$

when $x \geq \frac{1}{1+\lambda}$, for given initial value, and

$$\begin{aligned} \dot{x}_1 &= -\lambda x_1 \\ \dot{x}_2 &= x_2 \end{aligned}$$

when $0 \leq x \geq \frac{1}{1+\lambda}$. Then for initial value $p := (\frac{1}{2}, \frac{1}{2})$, $x_2 = \frac{1}{2}e^t$ which approaches infinity as $t \to \infty$. For x, when it reaches $x_1 = \frac{1}{1+\lambda}$ at some finite time t_0, it follows $\dot{x}_1 = -\lambda x_1$, hence the solution from t_0 onwards will be

$$x_1 = \frac{1}{1+\lambda} e^{-\lambda(t-t_0)}.$$

This means x approaches zero very slowly as $t \to \infty$. So the flow is a smooth pass approximating Dey's flow while there are not any new constructions as long as the lower transform is known. This shows that for initial data away from medial axis, the trajectory given by Dey's approach does not necessarily remain inside a region where $\text{dist}^2(x, K)$ is smooth, because the trajectory hits the medial axis in finite time.

Furthermore, the approach for surface reconstruction in [6] has considered the stable manifolds of index 2 non-degenerate critical points and has not treated the degenerate critical points case but we have also included this in our framework. Thus our approach has the advantage over the method of Dey, that degenerate critical points can also be used in surface reconstruction. We start by showing that the solution of the first order gradient system $\dot{x} = DC^l_\lambda \text{dist}^2(\cdot, K)$ is attracted to the critical point of $C^l_\lambda \text{dist}^2(\cdot, K)$ for the initial data in the interior of $C(K)$.

5.2 The gradient flow of the lower transform of squared distance function to a finite set in 3D

We investigate how the gradient flow, given by the solution to the first order gradient system of the lower transform $C^l_\lambda \text{dist}^2(x, K)$ of the squared distance function to a finite set $K \subset \mathbb{R}^n$ behaves for every initial data in the relative interior and in the relative boundary of the convex hull $C(K)$. In particular, we prove under some technical assumptions that for a finite set $K \subset \mathbb{R}^3$, the solution of the first order gradient system of the lower transform $C^l_\lambda \text{dist}^2(x, K)$ approaches the critical points of the lower transform of the squared distance function to the finite set K for every initial data in the relative interior and in the relative boundary of $C(K)$ as $t \to \infty$. We will illustrate the direction of the gradient flow for different examples of the lower transform $C^l_\lambda \text{dist}^2(x, K)$ of the squared distance functions to finite subsets of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

We provide some notation which will be used repeatedly in this section for convenience. Let us denote a sphere $\mathbb{S}^2(0, r)$ defined in Chapter 3 centred at origin $0 \in \mathbb{R}^3$ with radius $r > 0$ by $\mathbb{S}(0, r)$ or in general $\mathbb{S}$, a finite set $K \subset \mathbb{R}^3$ and subset $K(x)$ of K defined in 2.1.4, the relative interior of $C\big(K(x)\big)$ is denoted by ri $C\big(K(x)\big)$, the boundary of $C\big(K(x)\big)$ is denoted by $\partial C\big(K(x)\big)$ and x_0 is the geometric critical point unless stated otherwise. The lower transfrom of the squared distance function to the finite set K from Lemma 3.1.2 for critical point 0 is given by

$$F(x) := \frac{1}{2}C_\lambda^l \mathrm{dist}^2\big(x, K(0)\big) = \frac{1}{2}\Big\{(1+\lambda)\mathrm{dist}^2\Big(x, \frac{C(K(0))}{1+\lambda}\Big) + \frac{\lambda}{1+\lambda}r^2 - \lambda|x|^2\Big\}. \quad (5.2.2)$$

Furthermore, let us suppose that x_λ denotes the convex projection of x on $\frac{C(K(0))}{1+\lambda}$ and hence from [18, page 181],

$$\frac{1}{2}D\mathrm{dist}^2\Big(x, \frac{C\big(K(0)\big)}{1+\lambda}\Big) = x - x_\lambda, \quad (5.2.3)$$

where D is the gradient operator. The following is an interesting result concerning critical point of the lower transform $F(x)$ of the squared distance function to the finite set K.

Lemma 5.2.1. *Suppose that $0 \in \mathbb{R}^n$ be a geometric critical point of the lower transform $F(x)$ for the squared distance function to a finite set $K \subset \mathbb{R}^n$, that is, $0 \in C(K(0))$ and $K = K(0)$. Then 0 is the maximum point of F in $C(K(0))$.*

Proof. Suppose that $x \in C(K(0))$ and $x \neq 0$, then from (5.2.2) we have,

$$F(x) \leq F(0) \iff (1+\lambda)\mathrm{dist}^2\Big(x, \frac{C(K(0))}{1+\lambda}\Big) < \lambda|x|^2$$

Since $x \in C(K(0))$ and so $\frac{x}{1+\lambda} \in \frac{C(K(0))}{1+\lambda}$. Thus

$$\begin{aligned}(1+\lambda)\mathrm{dist}^2\Big(x, \frac{C(K(0))}{1+\lambda}\Big) &< (1+\lambda)\Big|x - \frac{x}{1+\lambda}\Big|^2 = \frac{\lambda}{1+\lambda}\lambda|x|^2 \\ &< \lambda|x|^2.\end{aligned}$$

Hence it follows that 0 is the maximum point of $F(x)$ in $C(K(0))$. □

We know from Definition 3.2.2 of the non-degenerate critical points that for x to be a non-degenerate critical point, x should be a relative interior point of $C(K(x))$. Therefore, we give the following result that will show that there is always a unique critical point of the lower transform $F(x)$ of the squared distance function to the finite set K in the relative interior of $C(K(x))$.

Lemma 5.2.2. *Suppose $K = K(0)$, and $0 \in ri\ C(K(0))$. Then $x = 0$ is the only critical point of F in $ri\ C(K(0))$.*

Proof. Suppose that x_λ be the projection of x onto $\frac{C(K(0))}{1+\lambda}$, then from (5.2.3), we have

$$DF(x) = x - (1+\lambda)x_\lambda.$$

Note that x is a critical point of F if and only if

$$x = (1+\lambda)x_\lambda.$$

If $x \in \frac{C(K(0))}{1+\lambda}$, $x = x_\lambda$, so

$$x = (1+\lambda)x_\lambda \iff x = 0.$$

If $x \in \text{ri } C(K(0)) \setminus \frac{C(K(0))}{1+\lambda}$, then x_λ belongs to the relative boundary of $\frac{C(K(0))}{1+\lambda}$, so $(1+\lambda)x_\lambda$ belongs to the relative boundary of $C(K(0))$, and thus it cannot equal $x \in \text{ri } C(K(0))$. Hence the only critical point of F in ri $C(K(0))$ is $x = 0$. □

In the case of non-degenerate critical points, we require the following result concerning the gradient flow of the lower transform $F(x)$ for the squared distance function to the finite set K. We prove that the solution of the gradient of the lower transform $DF(x)$ with initial condition lies in the relative interior of $C\big(K(0)\big)$ converge to the unique critical point 0 of the lower transform $F(x)$ for the squared distance function to the finite set K.

Lemma 5.2.3. *Suppose that $K = K(0)$, $0 \in ri\ C(K(0))$, let $u_0 \in ri\ C(K(0))$, and denote by $x(t)$ the solution of*

$$\begin{aligned} \dot{x}(t) &= DF(x(t)), \\ x(0) &= u_0. \end{aligned} \tag{5.2.4}$$

Further, let x_λ denote the convex projection of x on $\frac{C(K(0))}{1+\lambda}$ and assume,
(A1): If x is a point on the relative boundary of $\gamma C(K(0))$ where $\gamma \in \big(\frac{1}{1+\lambda}, 1\big]$, such that $x \in \gamma C(K^*)$ for some set $K^* = \{k_1, ..., k_m\} \subset K(0)$, then $x_\lambda \in \frac{C(K^*)}{1+\lambda}$.
Then there exists $\hat{T} \geq 0$ such that $x(\hat{T}) \in \frac{C(K(0))}{1+\lambda}$.

Remark: We believe that assumption (A1) is not needed, however at this stage we are not sure to exclude it.

Proof. The result is immediate if $u_0 \in \frac{C(K(0))}{1+\lambda}$. If we suppose that $u_0 \in \text{ri}\, C\big(K(0)\big)\setminus\frac{C(K(0))}{1+\lambda}$ and let $\gamma \in \big(\frac{1}{1+\lambda}, 1\big)$ be such that u_0 belongs to the relative boundary of $\gamma C(K(0))$. We first show that $x(t) \in \gamma C(K(0))$ for all $t \geq 0$. We know from Lemma 5.2.2 that

$$DF(x) = x - (1+\lambda)x_\lambda.$$

Since $(1+\lambda)x_\lambda$ belongs to the relative boundary of $C(K(0))$ whenever x_λ belongs to the relative boundary of $\frac{C(K(0))}{1+\lambda}$, the vector $x - (1+\lambda)x_\lambda$ points into the set $C(K(0))$ whenever $x \in \text{ri}\, C(K(0))$. Moreover, it follows from assumption (A1) that if x is a relative boundary point of $\gamma C(K(0))$ such that $x \in \gamma C(K^*)$ for some $K^* = \{k_1, ..., k_m\} \subset K(0)$, then $(1+\lambda)x_\lambda \in C(K^*)$, and so for each x in the relative boundary of $\gamma C(K(0))$, the 'face' of $\gamma C(K(0))$ to which x belongs is parallel to the 'face' of $C(K(0))$ containing $(1+\lambda)x_\lambda$. Thus the fact that the vector $x - (1+\lambda)x_\lambda$ points into $C(K(0))$ yields that its translate starting at x points into the set $\gamma C(K(0))$. Hence the flow $x(t)$ cannot leave the set $\gamma C(K(0))$ if u_0 belongs to the relative boundary of $\gamma C(K(0))$, and thus $x(t) \in \gamma C(K(0)) \subset \text{ri}\, C(K(0))$ for all $t \geq 0$.

Note from Lemma 5.2.2 that $x = 0$ is only critical point of F in ri $C(K(0))$, then for all $t > 0$,

$$|\dot{x}|^2 = DF(x(t)) \cdot \dot{x}(t) = \frac{d}{dt}F(x(t)),$$

and hence

$$F(x(t)) - F(x(0)) = \int_0^t |\dot{x}(s)|^2 ds. \tag{5.2.5}$$

Since $x(t) \in \gamma C(K(0))$ for all t, the left-hand side of equation (5.2.5) is bounded independently of t, whereas the right-hand side of equation (5.2.5) is a non-decreasing function of t. Hence there exists an increasing sequence $t_k \to \infty$ such that $\dot{x}(t_k) \to 0$. Taking a subsequence if necessary, we can ensure that there exists $x(t_k) \in \gamma C(K(0)) \subset \text{ri}\, C(K(0))$ such that

$$DF(x(t_k)) = \dot{x}(t_k) \to 0, \quad x(t_k) \to x \text{ as } t_k \to \infty.$$

But then $DF(x) = 0$, because $DF(\cdot)$ is continuous, so $x = 0$, since this is the only critical point of F in ri $C(K(0))$. Thus $x(t_k) \to 0$ as $t_k \to \infty$, and hence $x(t_k) \in \frac{C(K(0))}{1+\lambda}$ when k is sufficiently large. $\square$

Remark 5.2.4. The assumption (A1) holds for triangles in 2D, tetrahedra (simplices) in 3D, and note that this gives a local surface reconstruction result under the assuption of the existence of a regular Delaunay triangulation (in 2D) and Delaunay decomposition in 3D. The following remark explain how Assumption (A1) holds in 2D and 3D.

Remark 5.2.5. Let x be a point on the relative boundary of $\gamma C(K(0))$ where $\gamma \in \left(\frac{1}{1+\lambda}, 1\right]$, such that $x \in \gamma C(K^*)$ for some set $K^* = \{k_1, k_2\} \subset K(0) = \{k_1, k_2, k_3\}$ in $\mathbb{R}^2$ or $K^* = \{k_1, k_2, k_3\} \subset K(0) = \{k_1, k_2, k_3, k_4\}$ in $\mathbb{R}^3$, then $x_\lambda \in \frac{C(K^*)}{1+\lambda}$.

Suppose x is a point on the relative boundary of $\gamma C(K(0))$ where $\gamma \in \left(\frac{1}{1+\lambda}, 1\right]$. The shaded area in Figure 5.1 is $\gamma C(K(0)) \setminus \frac{C(K(0))}{1+\lambda}$ where $K^* = \{k_1, k_2\} \subset K$.

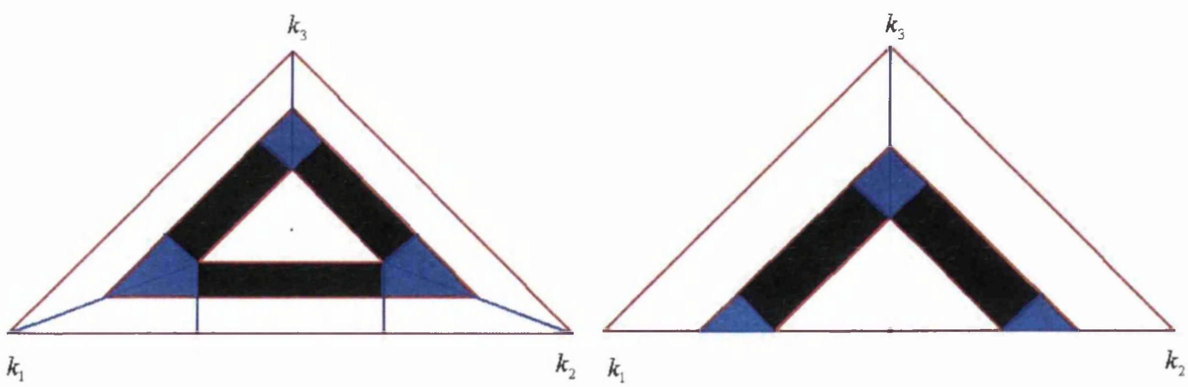

Figure 5.1: From left to right, triangle with non-degenerate critical point and degenerate critical point $0 \in \mathbb{R}^2$. The outermost triangle and its interior is $C(K(0))$, the smaller triangle of $C(K(0))$ and its interior is $\gamma C(K(0))$ for $\gamma \in \left(\frac{1}{1+\lambda}, 1\right]$, and the final triangle and its interior is $\frac{C(K(0))}{1+\lambda}$.

Clearly we note from Figure 5.1, that in the blue area the closest points of $\frac{C(K(0))}{1+\lambda}$ to the points of $\frac{K^*}{1+\lambda}$ is $\frac{k_1}{1+\lambda}$ and on the black area the closest points lies on $\frac{C(K^*)}{1+\lambda}$. Thus, for x on the relative boundary of $\gamma C(K^*)$, the convex projection $x_\lambda \in \frac{C(K^*)}{1+\lambda}$. Similarly, we can infer that this happens for the other subsets $\{k_1, k_3\}$ and $\{k_2, k_3\}$ of K by symmerty arguments. Hence, for x on the relative boundary of $\gamma C\big(K(0)\big)$, the convex projection $x_\lambda \in \frac{C(K(0))}{1+\lambda}$.

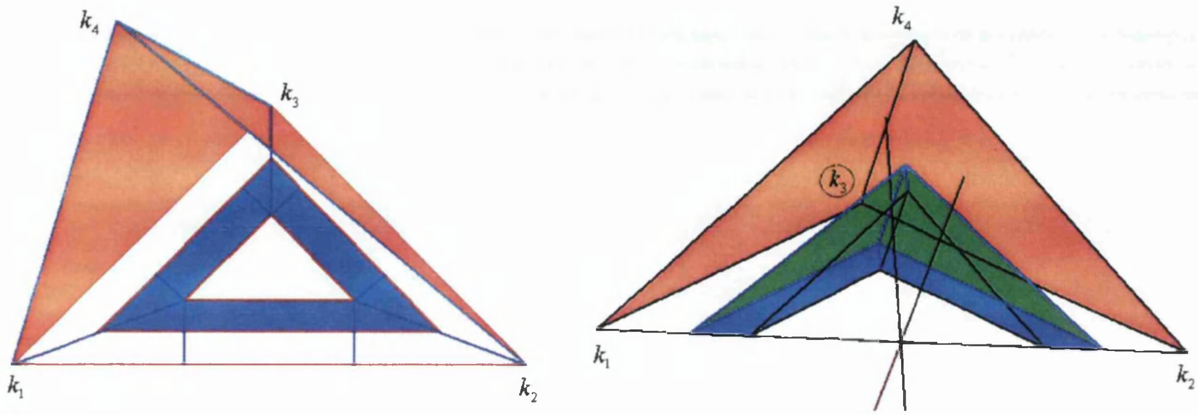

Figure 5.2: From left to right, tetradehra with the non-degenerate critical point and the degenerate critical point $0 \in \mathbb{R}^3$ where Both the tetrahedra are union of similar triangles as in 2D

Similarly, we note that in the following Figure 5.2, the tetrahedron are the union of similar triangles with $K^* = \{k_1, k_2, k_3\} \subset K = \{k_1, k_2, k_3, k_4\}$ as described in 2D in Figure 5.1 and hence for these triangles we infer the behaviour from 2D. It cannot be easily seen that interior of $\frac{K^*}{1+\lambda}$ has the closest point lies on $\frac{C(K^*)}{1+\lambda}$, hence we explain this in detail later by explicit examples.

We know from Definition 3.2.3 of the degenerate critical points that for x_0 to be a degenerate critical point, x_0 should be a relative boundary point of $C(K(x_0))$. Hence we need the following result.

Lemma 5.2.6. *Suppose $K = K(0)$ and 0 belongs to the relative boundary of $C(K(0))$. Suppose $\hat{K} := \{k_1, \ . \ . \ . \ , k_l\} \subset K(0)$ is such that $0 \in ri\ C(\hat{K})$. Assume (A1), then the only critical point of F in $ri\ C\big(K(0)\big) \cup ri\ C(\hat{K})$ is $x = 0$.*

Proof. Let x_λ be the projection of x onto $\frac{C(K(0))}{1+\lambda}$, then from (5.2.3) we have,

$$DF(x) = x - (1+\lambda)x_\lambda.$$

Note that x is a critical point of F if and only if

$$x = (1+\lambda)x_\lambda.$$

If $x \in \frac{C(K(0))}{1+\lambda}$, $x = x_\lambda$, then

$$x = (1+\lambda)x_\lambda \iff x = 0.$$

If $x \in \text{ri } C(K(0))$, then proof of Lemma 5.2.2 implies that x is not a critical point of F. If $x \in C(\hat{K})$ where $\hat{K} \subset K(0)$ such that $0 \in C(\hat{K})$ and $x \notin \frac{C(K(0))}{1+\lambda}$, then since (A1) holds, we know that if $x \in \text{ri } C(\hat{K})$, then

$$\text{dist}^2\Big(x, \frac{C(K(0))}{1+\lambda}\Big) = \text{dist}^2\Big(x, \frac{C(\hat{K})}{1+\lambda}\Big).$$

Hence, Lemma 5.2.2 implies that $x = 0$ is the only critical point of F in the relative interior of $C(\hat{K})$ and thus in the ri $C(K(0)) \cup \text{ri } C(\hat{K})$. □

Remark 5.2.7. A consquence of Lemma 5.2.6 is that for each $\gamma \in (0,1)$, the only critical point of F in $\gamma C(K(0))$ is $x = 0$.

In the case of degenerate critical points, we require the following result concerning the gradient flow of the lower transform $F(x)$ for the squared distance function to the finite set K. We show that the solution of the gradient of the lower transform $DF(x)$ with initial condition lies on the relative boundary of $\gamma C(K(0)) \subset C(K(0))$ for $\gamma \in (0,1)$ converges to the unique critical point of the lower transform F.

Lemma 5.2.8. *Suppose that $K = K(0)$, let 0 be in the relative boundary of $C(K(0))$, and $u_0 \in$ ri $C(K(0))$, and denote by $x(t)$ the solution of (5.2.4). Furthermore, let x_λ denote the convex projection of x on $\frac{C(K(0))}{1+\lambda}$ and assume (A1). Then there exists $\hat{T} \geq 0$ such that $x(\hat{T}) \in \frac{C(K(0))}{1+\lambda}$.*

Proof. The result is immediate if $u_0 \in \frac{C(K(0))}{1+\lambda}$. If we suppose that $u_0 \in \text{ri } C(K(0)) \setminus \frac{C(K(0))}{1+\lambda}$ and let $\gamma \in \big(\frac{1}{1+\lambda}, 1\big]$ be such that u_0 belongs to the relative boundary of $\gamma C(K(0))$. We first show that $x(t) \in \gamma C(K(0))$ for all $t \geq 0$. We know from Lemma 5.2.6 that

$$DF(x) = x - (1+\lambda)x_\lambda.$$

Since $(1+\lambda)x_\lambda$ belongs to the relative boundary of $C(K(0))$ whenever x_λ belongs to the relative boundary of $\frac{C(K(0))}{1+\lambda}$, the vector $x - (1+\lambda)x_\lambda$ points into the set $C(K(0))$

whenever $x \in \mathrm{ri}\, C\big(K(0)\big)$. Moreover, it follows from assumption (A1) that if x is a relative boundary point of $\gamma C\big(K(0)\big)$ such that $x \in \gamma C(K^*)$ for some $K^* = \{k_1, ..., k_m\} \subset K(0)$, then $(1+\lambda)x_\lambda \in C(K^*)$, and so for each x in the relative boundary of $\gamma C\big(K(0)\big)$, the 'face' of $\gamma C\big(K(0)\big)$ to which x belongs is parallel to the 'face' of $C(K(0))$ containing $(1+\lambda)x_\lambda$. Thus the fact that the vector $x - (1+\lambda)x_\lambda$ points into $C(K(0))$ yields that its translate starting at x points into the set $\gamma C\big(K(0)\big)$. Hence the flow $x(t)$ cannot leave the set $\gamma C(K(0))$ if u_0 belongs to the relative boundary of $\gamma C\big(K(0)\big)$, and thus $x(t) \in \gamma C\big(K(0)\big)$ for all $t \geq 0$.

Note from Lemma 5.2.6 that $x = 0$ is only critical point of F in the relative boundary of $\gamma C\big(K(0)\big)$, then for all $t > 0$,

$$|\dot{x}|^2 = DF(x(t)) \cdot \dot{x}(t) = \frac{d}{dt}F(x(t)),$$

and hence

$$F(x(t)) - F(x(0)) = \int_0^t |\dot{x}(s)|^2 ds. \tag{5.2.6}$$

Since $x(t) \in \gamma C\big(K(0)\big)$ for all t, the left-hand side of equation (5.2.6) is bounded independently of t, whereas the right-hand side of equation (5.2.6) is a non-decreasing function of t. Hence there exists a sequence $t_k \to \infty$ such that $\dot{x}(t_k) \to 0$. Taking a subsequence if necessary, we can ensure that there exists $x(t) \in \gamma C\big(K(0)\big)$ such that

$$DF(x(t_k)) = \dot{x}(t_k) \to 0, \quad x(t_k) \to x \text{ as } t_k \to \infty.$$

But we know that $x \in \gamma C\big(K(0)\big)$ and $DF(x) = 0$, so $x = 0$ since 0 is the only critical point of F in $\gamma C\big(K(0)\big)$ for each $\gamma \in (0,1)$. Thus $x(t_k) \to 0$ as $t_k \to \infty$, and hence $x(t_k) \in \frac{C(K(0))}{1+\lambda}$ when k is sufficiently large. Similar to the proof of Lemma 5.2.3, $x(t) \to 0$, as $t \to \infty$. $\square$

The following theorem concerning the gradient flow, given by the solution $x(t)$ to the first order gradient system of the lower transform $F(x)$ of the squared distance function to the finite set K, is easy to prove using Lemma 5.2.3 and 5.2.8.

Theorem 5.2.9. *Let $\hat{x} \in \mathbb{R}^3$ be a critical point of the lower transform $C_\lambda^l dist^2(x, K)$ of the squared distance function to a finite set $K \subset \mathbb{R}^3$, that is, $\hat{x} \in C\big(K(\hat{x})\big)$ and $K = K(\hat{x})$.*

Suppose $K(\hat{x})$ satisfies (A1) in Lemma 5.2.3. Then for every $x(0) = u_0$ in the relative interior of $C\big(K(\hat{x})\big)$ the solution $x(t)$ of system (5.2.4) is attracted to $\hat{x}$ as $t \to \infty$.

Proof. Let us suppose that $K \subset \mathbb{S}(0,r)$ be a finite set where $\mathbb{S}(0,r)$ is a sphere with radius $r > 0$ centered at 0. Assume for simplicity and without loss of generality, that $\hat{x} = 0 \in C\big(K(0)\big)$ and $K = K(0)$. We have to show that the solution $x(t)$ is attracted to 0 for the following two cases:

(i) when $u_0 \in \frac{C(K(0))}{1+\lambda}$,

(ii) when $u_0 \in \operatorname{ri} C\big(K(0)\big) \setminus \frac{C(K(0))}{1+\lambda}$.

In the first case (i), we suppose that $u_0 \in \frac{C(K(0))}{1+\lambda}$, the gradient points directly towards critical point 0, because in this case,

$$\dot{x}(t) = DF(x) = -\lambda x,$$

which implies that $DF(x)$ is always pointing directly towards 0, and this differential equation gives the solution $x(t) = u_0 e^{-\lambda t} \to 0$ as $t \to \infty$.

In the second case (ii), we suppose that $x \in \operatorname{ri} C\big(K(0)\big) \setminus \frac{C(K(0))}{1+\lambda}$. By Lemma 5.2.3 it follows that there exists $\hat{T} \geq 0$ such that $x(\hat{T}) \in \frac{C(K(0))}{1+\lambda}$, so it follows from the fact that the gradient $DF(x) = -\lambda x$ when $x \in \frac{C(K(0))}{1+\lambda}$ that for $t \geq \hat{T}$,

$$\dot{x}(t) = -\lambda x.$$

Hence solving this differential equation for $t \geq \hat{T}$ we have

$$x(t) = x(\hat{T}) e^{-\lambda(t-\hat{T})} \to 0 \quad \text{as} \quad t \to \infty.$$

This shows that the solution $x(t)$ of the system (5.2.4) is again attracted to 0 as $t \to \infty$ and hence the result follows. □

Remark 5.2.10. Theorem 5.2.9 can be extended to more general finite set K. In $C(K)$, if for every critical point $x_0 \in C(K)$, $\operatorname{dist}^2(x,K) = \operatorname{dist}^2\big(x, K(x_0)\big)$ for x in a neighbourhood of $C\big(K(x_0)\big)$ where $\lambda > 0$ is sufficiently large by using the Locality Theorem 4.1.1 and

Theorem 4.1.7. If K has a regular Delaunay triangulation, then if $x \in C\big(K(x_0)\big)$, we might have

$$C^l_\lambda \mathrm{dist}^2(x,K) = C^l_\lambda \mathrm{dist}^2\big(x, K(x_0)\big),$$

and we will explore this in future.

5.3 Non-degenerate index 1 Critical Point in 2D

We consider that for $x_0 \in \mathbb{R}^2$ be a geometric critical point of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to the finite set $K = K(x_0) \subset \mathbb{R}^2$, that is, $x_0 \in C\big(K(x_0)\big)$. We set the critical point as zero (i.e, $x_0 = 0$) and take $\dim C\big(K(0)\big) = 1$. Then, for $0 \in \mathbb{R}^2$ a relative interior point of $C\big(K(0)\big)$, from Theorem 3.2.7, point 0 is an non-degenerate index 1 critical point and $C\big(K(0)\big)$ is a one-dimenisonal convex set (an edge in this case).

Remark: Note that it is classified by Theorem 3.2.7 that if $\dim C\big(\big(K(0)\big)\big) = 1$, *then* 0 *is a non-degenerate index 1 critical point in* $\mathbb{R}^2$ *of the lower transform* $C^l_\lambda dist^2(x,K)$ *of the squared distance function to the finite set* $K \subset \mathbb{R}^2$.

We will show that the relative interior of $C(K)$ is in the stable manifold of a non-degenerate index 1 critical point and the boundary of $C(K)$ is in the stable manifold of non-degenerate index 0 critical point. Mathematically the stable manifold of non-degenerate index 1 critical point is

$$S(0) = \{y \in \mathbb{R}^2 : \text{ gradient flow of } C^l_\lambda \mathrm{dist}^2(y,K) \text{ converges to } 0 \text{ as } t \to \infty\}.$$

In other words, for every point as initial value in the relative interior of $C(K)$, the solution of the initial value problem

$$\begin{aligned} \dot{\underline{x}}(t) &= DC^l_\lambda \mathrm{dist}^2\big(x(t),K\big) \\ \underline{x}(0) &= u_0 \end{aligned} \tag{5.3.7}$$

approaches the non-degenerate index 1 critical point 0 as $t \to \infty$, that is, for x in the relative interior of $C(K)$ the direction of gradient flow coverges to non-degenerate index

1 critical point as $t \to \infty$. Hence in this case the edge (i.e., the relative interior of $C(K)$) is contained in the stable manifold of the non-degenerate index 1 critical point 0 and vertices are two points of K. Therefore, the curve reconstruction is given by the union of edge in the stable manifolds of non-degenerate index 1 critical points and vertices in the stable manifold of non-degenerate index 0 critical points. The following example will illustrate this in detail.

Example 5.3.1. Let us consider the lower transform $h(x,y) := C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ of the squared distance function to finite set $K = \big\{(0,1),(0,-1)\big\}$ from Example (3.8), that is,

$$h(x,y) = \begin{cases} x^2 + (y-1)^2 & y > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + x^2 - \lambda y^2 & -\frac{1}{1+\lambda} \le y \le \frac{1}{1+\lambda} \\ x^2 + (y+1)^2 & y < -\frac{1}{1+\lambda}. \end{cases}$$

The critical point $x_0 := (0,0)$ is in the relative interior of $C\big(K(x_0)\big)$ for $K(x_0) = K$ and $\dim C(K) = 1$, that is, the relative interior of $C(K)$ is a 1-dimensional convex set (an edge for such finite set K) and its relative boundary consists of points $(0,1)$ and $(0,-1)$. This implies that the point x_0 is a non-degenerate index 1 critical point of the lower transform of the squared distance function to the set K.

The behaviour of the solution of initial value problem (5.3.7) of the lower transform when intial data lies inside the relative interior of $C(K)$.

Our aim is to understand the behaviour of the solution $\underline{x}(t) = \big(x(t),y(t)\big)$ of the initial value problem (5.3.7), when the initial value $u_0 = \big(0,y(0)\big)$ is in the relative interior of $C(K)$. In fact, we show that when u_0 is in the relative interior of $C(K)$, the solution $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. The solution $\underline{x}(t)$ does not converge to the non-degenerate index 1 critical point x_0, when u_0 lies in the relative boundary of $C(K)$, which in fact equals K in this case. In fact, if $u_0 \in K$, then clrealy $x(t) = u_0$ for all t since points of K are critical points of $C^l_\lambda \mathrm{dist}^2(\cdot,K)$.

Remark: Note that an explicit formula for the solution $x(t)$ of the first order gradient system of $C^l_\lambda\, dist^2\big(x,K\big)$ in terms of initial conditions $x(0)$ has been computed in Appendix

B (B.1.2). Therefore, by replacing x with y we can easily get our required solution $\underline{x}(t) = \big(x(t),\ y(t)\big)$. Note that $x(t) = 0$ for all t in this example.

Let us suppose that the initial condition u_0 lies inside $\frac{C(K(x_0))}{1+\lambda}$. Then the solution we get is given by

$$\begin{aligned} x(t) &= 0 \\ y(t) &= y(0)e^{-2\lambda t}. \end{aligned}$$

This solution will tend to the critical point x_0 as $t \to \infty$ beacuse the term $e^{-2\lambda t} \to 0$ as $t \to \infty$. Furthermore, note that, the gradient in this region,

$$Dh(x, y) = \begin{pmatrix} 0 \\ -2\lambda y \end{pmatrix}$$

always points directly towards x_0, showing that the direction of the gradient flow of the lower transform $h(x, y)$ of the squared distance function to the finite set K is always directly towards the non-degenerate index 1 critical point x_0.

Again, suppose that $\frac{1}{1+\lambda} < y(0) < 1$, then the solution is given by

$$y(t) = (y(0) - 1)e^{2t} + 1.$$

If $0 < y(0) < 1$, then $\dot{y}(t) = 2(y(0) - 1)e^{2t} < 0$. This implies that the solution $y(t)$ is decreasing and thus will enter $\frac{C(K(x_0))}{1+\lambda}$ and so by above arguments the solution $\underline{x}(t)$ will approach the non-degenerate index 1 critical point x_0. Furthermore, we note that when $y < 1$ the gradient,

$$Dh(x, y) = \begin{pmatrix} 0 \\ 2(y - 1) \end{pmatrix}$$

is pointing towards $\frac{C(K(x_0))}{1+\lambda}$ and hence towards x_0. This shows that the direction of the gradient flow of the lower transform of the squared distance function to finite set K converges to the non-degenerate index 1 critical point as $t \to \infty$. Analagously, suppose that $-1 < y(0) < -\frac{1}{1+\lambda}$, then by similar arguments as above the solution $\underline{x}(t)$ tends to x_0 as $t \to \infty$ and the gradient of the lower transform $h(x, y)$ converges to the non-degenerate index 1 critical point x_0 as $t \to \infty$.

The behaviour of the solution of initial value problem (5.3.7) of the lower transform when intial data in the relative boundary of $C(K)$.

We try to understand the behaviour of the solution $\underline{x}(t)$ in the relative boundary of $C(K)$. We note that in this example the relative boundary of convex hull of K is just the set of two points $(0,1)$ and $(0,-1)$. Therefore, when the initial condition u_0 belongs to K (i.e., on the relative boundary points of $C(K)$), then on the point $(0,1)$ the solution $x(t)=0$ and $y(t)=1$ and for the point $(0,-1)$ the solution $x(t)=0$ and $y(t)=-1$ for all t. We also note that in these relative boundary points $Dh(x,y)=0$ because, for $(0,1)$

$$Dh(x,y)=\begin{pmatrix}0\\2(y-1)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

and the point $(0,-1)$

$$Dh(x,y)=\begin{pmatrix}0\\2(y+1)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

This implies that $(0,-1)$ and $(0,1)$ are critical points of the lower transform $h(x,y)$, and in particular, the gradient does not converge to x_0 as $t\to\infty$. Thus when the initial conditions lie in the relative boundary of $C(K)$, the solution $\underline{x}(t)$ does not converge to the non-degenerate index 1 critical point x_0, but converges (in fact, is equal) to the non-degenerate index 0 critical points $(0,1)$ and $(0,-1)$.

Hence we have shown that the edge (i.e., relative interior of $C(K)$ in this example) is in the stable manifold of the non-degenerate index 1 critical point, whereas the boundary points of $C(K)$ are in the stable manifold non-degenerate index 0 critical points. Therefore, the curve reconstruction in this example is the union of interior of $C(K)$ in the stable manifold of non-degenerate index 1 critical point x_0 and stable manifolds of non-degenerate index 0 critical points $(0,1)$ and $(0,-1)$.

5.4 Non-degenerate index 2 Critical Point in 3D

Let $x_0\in\mathbb{R}^3$ be a geometric critical point of the lower transform of the squared distance function to finite set $K=K(x_0)\subset\mathbb{R}^3$, that is, $x_0\in C\big(K(x_0)\big)$. We set the critical point as zero (i.e, $x_0 = 0$) and take $\dim C\big(K(0)\big) = 2$. Then, for $0 \in \mathbb{R}^3$ a relative interior point of $C\big(K(0)\big)$ from Theorem 3.2.9, the point 0 is a non-degenerate index 2 critical point and $C\big(K(0)\big)$ is a two-dimenisonal convex polygon (a triangle in this case).

Remark: Note that it is classified in Chapter 3 *Theorem 3.2.9 that if* $\dim C\big(K(0)\big) = 2$, *then* 0 *is a non-degenerate index 2 critical point in* $\mathbb{R}^3$ *of the lower transform of the squared distance function to the finite set* K.

We will show that the relative interior of $C(K)$ is in the stable manifold of a non-degenerate index 2 critical point and the relative boundary of $C(K)$ is in the stable manifolds of non-degenerate index 1 and index 0 critical points. Mathematically the stable manifold of non-degenerate index 2 critical point in $\mathbb{R}^3$ is

$$S(0) = \{y \in \mathbb{R}^3 : \text{ gradient flow of } C^l_\lambda \mathrm{dist}^2(y, K) \text{ converges to } 0 \text{ as } t \to \infty\}.$$

In other words, for every point as initial value in the relative interior and relative boundary of $C(K)$ the solution of the initial value problem

$$\begin{aligned} \dot{\underline{x}}(t) &= DC^l_\lambda \mathrm{dist}^2\big(\underline{x}(t), K\big) \\ \underline{x}(0) &= u_0 \end{aligned} \tag{5.4.8}$$

approaches the non-degenerate index 2 critical point 0 and non-degenerate index 1 and index 0 critical points as $t \to \infty$ respectively, that is, in the relative interior of $C(K)$ the gradient flow converges to non-degenerate index 2 critical point as $t \to \infty$, but on relative boundary of $C(K)$ the gradient flow converges to non-degenerate index 1 and index 0 critical points. This implies that the relative interior of the convex polygon is contained in the stable manifold of the non-degenerate index 2 critical point x_0 and the relative boundary of $C(K)$ is contained in stable manifolds of non-degenerate index 1 and index 0 critical points. Therefore, a surface reconstruction is given by the union of the open polygons given by the stable manifolds of non-degenerate index 2 critical points, the relative edges in stable manifold of non-degenerate index 1 critical points and points of K in the stable manifold of non-degenerate index 0 critical points. The following example will illustrate this idea in detail.

Example 5.4.1. Let us consider the Example (4.2.3) where the lower transform of the squared distance function to the finite set $K = \left\{(0,1,0), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right)\right\}$ from Chapter 4 is given by

$$h(x,y,z) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2) & |x| \geq \frac{1}{\sqrt{3}}\left(y - \frac{1}{1+\lambda}\right), y \geq \frac{-1}{2(1+\lambda)} \\ x^2 + (y-1)^2 + z^2 & |x| \leq \sqrt{3}\left(y - \frac{1}{1+\lambda}\right), y > \frac{1}{1+\lambda} \\ \left(x + \frac{\sqrt{3}}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + z^2 & x < \frac{-\sqrt{3}}{2(1+\lambda)}, y \leq \frac{-1}{\sqrt{3}}x - \frac{1}{1+\lambda} \\ \left(x - \frac{\sqrt{3}}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + z^2 & x > \frac{\sqrt{3}}{2(1+\lambda)}, y \leq \frac{1}{\sqrt{3}}x - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y - \frac{1}{2(1+\lambda)}\right)^2 - \lambda(x^2+y^2) + z^2 & \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)} < x < \frac{\sqrt{3}(1+y(1+\lambda))}{1+\lambda}, \\ & y < \frac{1}{\sqrt{3}}x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \left(\frac{-\sqrt{3}}{2}x + \frac{1}{2}y - \frac{1}{2(1+\lambda)}\right)^2 - \lambda(x^2+y^2) + z^2 & \frac{\sqrt{3}(1+y(1+\lambda))}{1+\lambda} < x < \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)}, \\ & y < -\frac{1}{\sqrt{3}}x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \left(y + \frac{1}{2(1+\lambda)}\right)^2 - \lambda(x^2+y^2) + z^2 & |x| < \frac{\sqrt{3}}{2(1+\lambda)}, y < -\frac{1}{2(1+\lambda)} \end{cases}$$

The lower transform for different regions specified in the above expression is denoted by the regions from the start by regions $(*)$, (1), (2), (3), (4), (5) and region (6) respectively, as shown in Figure 5.3. The critical point $x_0 := (0,0,0)$ is a relative interior point of $C(K(x_0))$ for $K(x_0) = K$ and $\dim C(K) = 2$, that is, the relative interior of $C(K)$ is a 2-dimensional convex polygon (triangle for such finite set K) and the relative boundary is the points of K and relative edges between these points. This implies that the critical point x_0 is a non-degenerate index 2 critical point of the lower transform of the squared distance function to the finite set K.

The behaviour of the solution of the initial value problem (5.4.8) of the lower transform when intial condition lies inside the relative interior of $C(K)$.

Our aim is to understand the behaviour of the solution $\underline{x}(t) = \big(x(t), y(t), z(t)\big)$ of the initial value problem (5.4.8) when the initial value $u_0 = \big(x(0), y(0), 0\big)$, is in the relative interior of $C(K)$. Hence we show that $x(t) \to 0$, $y(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$, that is, the gradient flow converges to non-degenerate index 2 critical point. Figure 5.3 shows symmetry between regions in the relative interior of $C(K)$ and so we will solve one of the symmetric regions and infer the behaviour in the other regions by symmetry arguments.

Remark: Note that the explicit formula for the solution $x(t)$ and $y(t)$ of the first order gradient system of $C_\lambda^l\big((x,y),K\big)$ in terms of initial conditions $\big(x(0),y(0),0\big)$ has been calculated in Appendix B (B.1.5) and (B.1.6) respectively. We can therefore extend it easily to the solution $\underline{x}(t) = \big(x(t),y(t),z(t)\big)$ of first order gradient system of $C_\lambda^l\big((x,y,z),K\big)$.

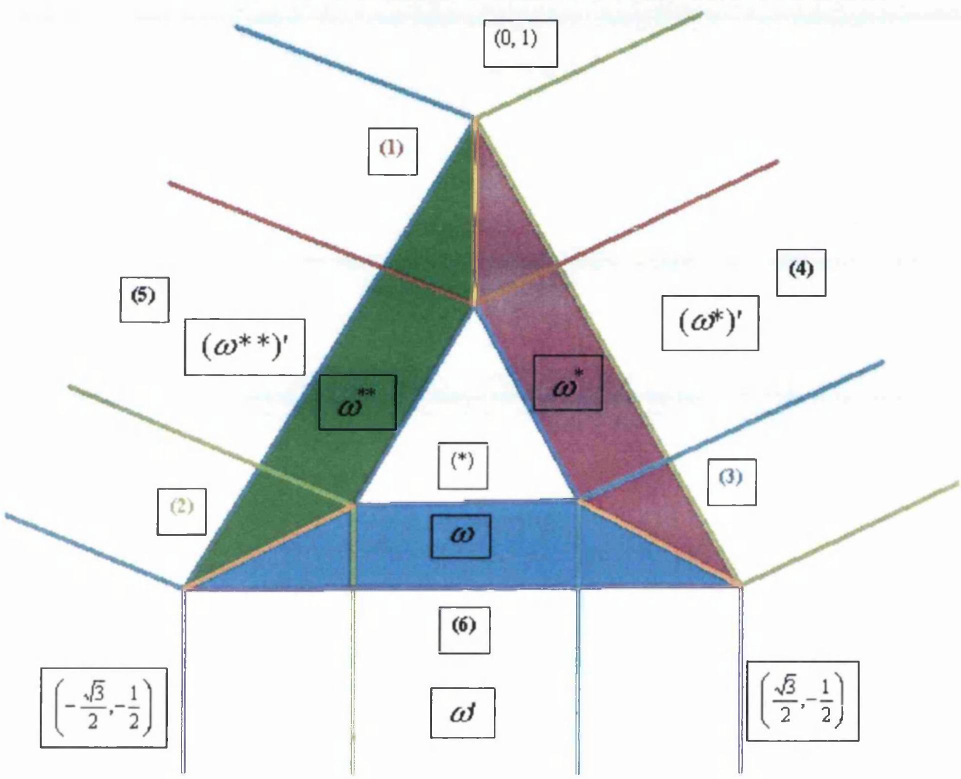

Figure 5.3: Symmetric regions of $C(K)$

We define regions ω, ω^* and ω^{**} subsets of the relative interior of $C\big(K\big)$ such that they are symmetric to each other as shown in Figure 5.3. The region ω is defined such that it includes region (6), part of region (2) and (3) as shown in the Figure 5.3. The region ω^* is defined such that it contains region (4), part of region (1) and (3) symmetric to ω. Analogously region ω^{**} is defined such that it is a combination of region (5), part of region (1) and (2). Since these regions are symmetric, we show the behaviour of the solution $\underline{x}(t)$ only when the initial condition $\big(x(0),y(0),0\big)$ lies in region ω and can infer the results for other regions by symmetry arguments. In fact, we show that the solution $x(t) \to 0$, $y(t) \to 0$ and $z(0) \to 0$ as $t \to \infty$, that is, the solution $\underline{x}(t)$ tends to the non-degenerate index 2 critical point x_0 as $t \to \infty$. Note that in this particular example, $z(t) = 0$ for all time t, so we do not further investigate the behaviour of $z(t)$ and hence we will investigate only the behaviour of $x(t)$ and $y(t)$.

Suppose that the initial condition u_0 lies in region (*), that is, inside $\frac{C(K)}{1+\lambda}$. Then the solution from (B.1.5) and (B.1.6) gives

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \\ y(t) &= y(0)e^{-2\lambda t} \end{aligned}$$

We need to show that the solution will tend to x_0 as t tends to infinity. This can be seen clearly because the term $e^{-2\lambda t} \to 0$ as $t \to \infty$. Furthermore, note that, the gradient in this region,

$$DC_\lambda^l \text{dist}^2\big((x,y,z),K\big) = -2\lambda \begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$

always points towards x_0, showing that the direction of the gradient flow of the lower transform is always directly towards the non-degenerate index 2 critical point x_0.

The behaviour of the solution when initial condition lies inside the region ω. We first investigate that for the initial condition u_0 in region (6), the solution $\underline{x}(t)$ enters region (*) and thus by arguments of region (*) the solution $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. Then we show that in region (2), if the initial condition u_0 is inside the part of region (2) $\in \omega \subset C(K)$, then $x(t)$ is increasing. Finally we show that in region (3), if the initial conditions u_0 is inside the part of region (3) $\in \omega \subset C(K)$, then $x(t)$ is decreasing. Therefore, from both the regions the solution $\underline{x}(t)$ enters region (6) and from there it follows that $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$.

Suppose that the initial condition u_0 lies in region (6), that is, $|x(0)| \leq \frac{\sqrt{3}}{2(1+\lambda)}$ and

$y(0) < -\frac{1}{2(1+\lambda)}$. Then the solution from (B.1.5) and (B.1.6) gives

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \\ y(t) &= \big(y(0) + \frac{1}{2}\big)e^{2t} - \frac{1}{2}. \end{aligned}$$

We show that when u_0 lies in region (6), the solution $\underline{x}(t)$ will exit region (6) at some time t through the line $y = -\frac{1}{2(1+\lambda)}$.

It can be obviously seen that if we take $\frac{-\sqrt{3}}{2(1+\lambda)} \leq x(0) < 0$, then $x(t)$ is increasing and $x(0) < x(t) < 0$ whenever t is such that $\underline{x}(t)$ lies in region (6). Similarly, if we take $0 < x(0) \leq \frac{\sqrt{3}}{2(1+\lambda)}$, then $x(t)$ is decreasing and $0 < x(t) < x(0)$ whenever t is such that $\underline{x}(t)$ lies in region (6). Now we suppose $\frac{-1}{2} < y(0) \leq \frac{-1}{2(1+\lambda)}$ in region (6). If $y(0) > -\frac{1}{2}$, then

$$\dot{y}(t) > 2\big(y(0) + \frac{1}{2}\big) > 0$$

This implies that if $y(0)$ is in the relative interior of $C(K)$, then $y(t)$ is increasing and so $y(t)$ will intersect the line between region $(*)$ and region (6). Hence, the solution $\underline{x}(t)$ can only exit region (6) through the line $y = -\frac{1}{2(1+\lambda)}$, since $\underline{x}(t)$ cannot exit region (6) either through line $x = -\frac{\sqrt{3}}{2(1+\lambda)}$ or $x = \frac{\sqrt{3}}{2(1+\lambda)}$ as argument above for $x(t)$ in region (6). So the solution $\underline{x}(t)$ must enter region (*), from which it follows that the solution $\underline{x}(t)$ tends to the critical point x_0 as $t \to \infty$, as argued above in region (*). Furthermore, we calculate the gradient in region (6),

$$DC^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big) = \begin{pmatrix} -2\lambda x \\ 2y+1 \\ 0 \end{pmatrix}.$$

Note that at the boundary between region (*) and (6), that is, the line $y = \frac{-1}{2(1+\lambda)}$ the y-component of the gradient is positive,

$$\dot{y}(t) = 2y + 1 = \frac{1}{1+\lambda} > 0.$$

This implies that the gradient of the lower transform points towards $\frac{C(K)}{1+\lambda}$, and in fact, the gradient is pointing towards the non-degenerate index 2 critical point x_0.

Remark: Note that as $DC_\lambda^l dist^2((x,y,z),K)$ is Lipschitz continuous, since $C_\lambda^l dist^2((x,y,z),K)$ is $C^{1,1}$ and it points directly towards $\underline{x}(t)$ in region (*). Then it also points directly towards $\underline{x}(t)$ when it is on the boundary between region (*) and (6).

Now suppose that the initial condition u_0 lies in region (2) such that $x(0) < \frac{-\sqrt{3}}{2(1+\lambda)}$ and $y(0) \le \frac{1}{\sqrt{3}}x(0)$. Then the solutions from (B.1.5) and (B.1.6) are

$$\begin{aligned} x(t) &= \Big(x(0) + \frac{\sqrt{3}}{2}\Big)e^{2t} - \frac{\sqrt{3}}{2} \\ y(t) &= \Big(y(0) + \frac{1}{2}\Big)e^{2t} - \frac{1}{2}. \end{aligned}$$

By combining the two equations we get

$$y(t) = \Big(\frac{2y(0)+1}{2x(0)+\sqrt{3}}\Big)x(t) + \Big(\frac{\sqrt{3}y(0) - x(0)}{2x(0)+\sqrt{3}}\Big) \tag{5.4.9}$$

which passes through the point $(\frac{-\sqrt{3}}{2}, \frac{-1}{2})$. This shows that when the initial condition u_0 is inside the region (2) $\subset \omega$ the solution enters region (6) through the line $x = -\frac{\sqrt{3}}{2(1+\lambda)}$. Therefore, by arguing as above, we have shown that the solution $\underline{x}(t)$ will reach the boundary of region (*), and thus $\underline{x}(t)$ tends to x_0 as $t \to \infty$. Furthermore, we calculate the gradient in region (2),

$$DC_\lambda^l \mathrm{dist}^2((x,y,z),K) = \begin{pmatrix} 2x+\sqrt{3} \\ 2y+1 \\ 0 \end{pmatrix}.$$

Note that at the boundary between region (2) and (6), that is, the line $x = -\frac{\sqrt{3}}{2(1+\lambda)}$ the x-component of the gradient is positive,

$$\dot{x}(t) = 2x + \sqrt{3} = \frac{\sqrt{3}\lambda}{1+\lambda} > 0.$$

This implies that the gradient of the lower transform points towards region (6) and then by arguing as above points towards $\frac{C(K)}{1+\lambda}$, and in fact, the gradient is pointing towards x_0. Similarly, by symmerty arguments as in region (2), if the initial condition u_0 lies in region (3), then $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$ and the gradient of the lower transform points towards x_0. Analogously, for region ω^* and ω^{**}, the solution $\underline{x}(t)$ approaches x_0 as $t \to \infty$.

Hence we have shown that for the initial condition u_0 in the relative interior of $C(K)$ the solution $\underline{x}(t) := \big(x(t), y(t), z(t)\big)$ approaches x_0 as $t \to \infty$, that is, the gradient flow converges to the non-degenerate index 2 critical point x_0 as $t \to \infty$.

The behaviour of the solution of the initial value problem (5.4.8) of the lower transform when initial condition lies in the relative boundary of $C(K)$.

For this particular example the relative boundary of $C(K)$ consists of the points of K and the relative edges between these points. We have shown in our earlier Example (5.3.1) that relative edges are in the stable manifold of index 1 and points in the stable maniflods of index 0 critical points. Therefore, we only show that:

(i) The solution $\underline{x}(t)$ of the lower transform of the squared distance function to the set K on the relative edge between $\big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big)$ and $\big(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big)$ converges to $\big(0, -\frac{1}{2}, 0\big)$ when initial condition lies on this relative edge.

(ii) The solution $\underline{x}(t)$ of the lower transform of squared distance function to set K on the point $\big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big)$ converges to $\big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big)$ when initial condition lies on this point of the finite set K.

Let us consider the relative edge described in (i) and suppose that the initial condition lies on this particular edge of $C(K)$ in region (6), that is, $-\frac{\sqrt{3}}{2} < x(0) < \frac{\sqrt{3}}{2}$ and $y(0) = -\frac{1}{2}$, then we have

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \to 0 \text{ as } t \to \infty \\ y(t) &= \big(y(0) + \frac{1}{2}\big)e^{2t} - \frac{1}{2} = -\frac{1}{2} \qquad \forall\, t \end{aligned}$$

Furthermore, we note that for this relative edge the gradient points towards $\frac{C(K)}{1+\lambda}$ because the y-component of gradient is

$$\dot{y}(t) = 2(y+1) = -\frac{1}{2}.$$

Let us consider the point described in (ii) and so when $x(0) = -\frac{\sqrt{3}}{2}$ and $y(0) = -\frac{1}{2}$,

$$\begin{aligned} x(t) &= \Big(x(0) + \frac{\sqrt{3}}{2}\Big)e^{2t} - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \quad \forall\, t \\ y(t) &= \Big(y(0) + \frac{1}{2}\Big)e^{2t} - \frac{1}{2} = -\frac{1}{2} \quad \forall\, t \end{aligned}$$

The gradient,

$$DC^l_\lambda \mathrm{dist}^2\big((x,y,z),K\big) = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

always points into $\big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big)$. Therefore, when initial condition u_0 lies on the relative boundary of $C(K)$ the solution $\underline{x}(t)$ does not converge to the non-degenerate index 2 critical points but does converge to the non-degenerate index 1 and index 0 critical points on the relative boundary. Analogously, for other relative edges and points the solution converges to non-degenerate index 1 and index 0 critical points respectively.

Hence we have shown that the relative interior of the convex polygon (i.e., the relative interior of $C(K)$) is in the stable manifold of the non-degenerate index 2 critical point. We have also shown that the relative edges (i.e., relative interior of $\partial C(K)$) and points (i.e., relative boundary points of $C(K)$) are not in the stable manifold of the non-degenerate index 2 critical point, but are in the stable manifold of non-degenerate index 1 and index 0 critical points respectively. Thus the surface reconstruction is given by the union of the relative interior of the convex polygons of non-degenerate index 2 critical point and the curve reconstruction is given by the union of the relative edges of non-degenerate index 1 and points of non-degenerate index 0 critical points.

The behaviour of the solution of the initial value problem (5.4.8) of the lower transform when initial data lies outside $C(K)$.

In this part we will understand the behaviour of the solution of the initial value problem (5.4.8) when the initial value u_0 is outside $C(K)$. We will show that, the solution $x(t) \to \infty$ and $y(t) \to \infty$ as $t \to \infty$ and the gradient flow of the lower transform of squared distance function to finite set K does not converge to the non-degenerate index 2 critical point. We know from Figure 5.3 that regions ω, ω^* and ω^{**} are symmetric and contained in $C(K)$.

Therefore, we denote regions ω', $(\omega^*)'$ and $(\omega^{**})'$ as the regions that are not contained by $C(K)$. Note that we defined here only region ω' and the other two regions are symmetric and can be defined in similar fashion, as shown in the diagram.

$$\omega' := \left\{(x,y,z) \in \mathbb{R}^3 : \{|x| \leq \frac{\sqrt{3}}{2}, y < \frac{-1}{2}\} \cup \{x > \frac{\sqrt{3}}{2}, y \leq \frac{-1}{\sqrt{3}}x\} \cup \{x < \frac{-\sqrt{3}}{2}, y \leq \frac{1}{\sqrt{3}}x\}\right\}$$

The behaviour of the solution $\underline{x}(t)$ of the lower transform in region (6) when $y(0) < -\frac{1}{2}$ can be inferred from the argument above for region (6). This implies that, if the initial condtion u_0 does not lie inside $C(K)$ but lies in region ω', then the solution $x(t) \to 0$ and $y(t) \to -\infty$ as $t \to \infty$. Similarly, the gradient in region (6) shows that, if $y < -\frac{1}{2}$, then the y-component of the gradient is negative,

$$\dot{y}(t) = 2y + 1 = -\frac{1}{2} < 0.$$

This implies that the gradient of the lower transform points away from the $C\big(K(x_0)\big)$ and hence diverge from non-degenerate index 2 critical point x_0 as $t \to \infty$.

Further, suppose that the initial condition u_0 lies in region (2) such that $x(0) < \frac{-\sqrt{3}}{2}$ and $y(0) \leq \frac{1}{\sqrt{3}}x(0)$. Then by the same argument as in region (2) $\in \omega$, the solution will intersect line $x = \frac{-\sqrt{3}}{2}$ and enter region (6) for which $y(0) < -\frac{1}{2}$ and in fact, the solution $x(t) \to 0$ and $y(t) \to -\infty$ as $t \to \infty$ as argued in region (6) $\in \omega'$. The gradient in region (2),

$$Dh(x,y,z) = \begin{pmatrix} 2x + \sqrt{3} \\ 2y + 1 \\ 0 \end{pmatrix}$$

shows that, if $x < -\frac{\sqrt{3}}{2}$ and $y < -\frac{1}{2}$, then the x-component and y-component of gradient,

$$\begin{aligned} \dot{x}(t) &= 2x + \sqrt{3} < -\frac{\sqrt{3}}{2} < 0 \\ \dot{y}(t) &= 2y + 1 < -\frac{1}{2} < 0 \end{aligned}$$

implies that the gradient flow of the lower transform diverges from the $C\big(K(x_0)\big)$ as $t \to \infty$ and so from non-degenerate index 2 critical point x_0.

Analogously, if the initial value u_0 lies in region (3) such that $x(0) > \frac{\sqrt{3}}{2}$ and $y(0) \leq -\frac{1}{\sqrt{3}}x(0)$, then the solution $x(t) \to 0$ and $y(t) \to -\infty$ as $t \to \infty$ as argued in region

(6) $\in \omega'$. By similar arguments, the gradient of the lower transform in region (3) $\in \omega'$ points away from the $C\big(K(x_0)\big)$ and so from non-degenerate index 2 critical point x_0.

5.5 Non-degenerate index 3 Critical Points in 3D

We consider $x_0 \in \mathbb{R}^3$ to be a geometric critical point of the lower transform of the squared distance function to finite set $K = K(x_0) \subset \mathbb{R}^3$, that is, $x_0 \in C\big(K(x_0)\big)$. We set the critical point as zero ($x_0 = 0$) and take $\dim C(K) = 3$. Then, for $0 \in \mathbb{R}^3$ in the relative interior of $C(K)$ from Theorem 3.2.9 the critical point 0 is a non-degenerate index 3 critical point and the interior of $C(K)$ is a three-dimenison polytope.

Remark: Note that it is classified by Theorem 3.2.9 that if $\dim C\big(K(0)\big) = 3$, *then* 0 *is a non-degenerate index 3 critical point in* $\mathbb{R}^3$ *of the lower transform of the squared distance function to the finite set* K.

We have to show that the relative interior of $C(K)$ is in the stable manifold of a non-degenerate index 3 critical point and the relative boundary of $C(K)$ which in this example are faces of $C(K)$, edges of $C(K)$ and points of K, are in the stable manifolds of non-degenerate index 2, index 1 and index 0 critical points respectively. In other words, first we show that for every point as initial value in the relative interior of $C(K)$ the solution of the initial value problem

$$\begin{aligned} \dot{\underline{x}}(t) &= DC^l_\lambda \mathrm{dist}^2\big(\underline{x}(t), K\big) \\ \underline{x}(0) &= u_0 \end{aligned} \tag{5.5.10}$$

approaches the non-degenerate index 3 critical point x_0 as $t \to \infty$, that is, the gradient flow converges towards x_0 as $t \to \infty$. This means that the three-dimensional polytope, that is, the interior of $C(K)$ is contained in the stable manifold of the non-degenerate index 3 critical point.

The stable manifold of non-degenerate index 3 critical point is a three-dimensional open polytope which is bounded by the stable manifolds of non-degenerate index 2 (faces of $C(K)$), index 1 (edges of $C(K)$) and index 0 critical points (points of K). The surface reconstruction is given by the union of two-dimensional polygons in the stable manifolds of non-degenerate index 2 critical points, their edges and points of K in the stable manifolds of non-degenerate index 1 and 0 critcal points respectively.

Example 5.5.1. Let us consider Example 4.2.4

$$h = \begin{cases} \frac{3\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \left(|x|-1\right)^2 + \left(|y|-1\right)^2 + \left(|z|-1\right)^2 & |x| > \frac{1}{1+\lambda}, |y| > \frac{1}{1+\lambda}, |z| > \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\left(|y| - \frac{1}{1+\lambda}\right)^2 - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| > \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\left(|x| - \frac{1}{1+\lambda}\right)^2 - \lambda(x^2+y^2+z^2) & |x| > \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\left(|z| - \frac{1}{1+\lambda}\right)^2 - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| > \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\left(\left(|x| - \frac{1}{1+\lambda}\right)^2 + \left(|y| - \frac{1}{1+\lambda}\right)^2\right) - \lambda(x^2+y^2+z^2) & |x| \ge \frac{1}{1+\lambda}, |y| \ge \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\left(\left(|x| - \frac{1}{1+\lambda}\right)^2 + \left(|z| - \frac{1}{1+\lambda}\right)^2\right) - \lambda(x^2+y^2+z^2) & |x| \ge \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \ge \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\left(\left(|y| - \frac{1}{1+\lambda}\right)^2 + \left(|z| - \frac{1}{1+\lambda}\right)^2\right) - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \ge \frac{1}{1+\lambda}, |z| \ge \frac{1}{1+\lambda} \end{cases}$$

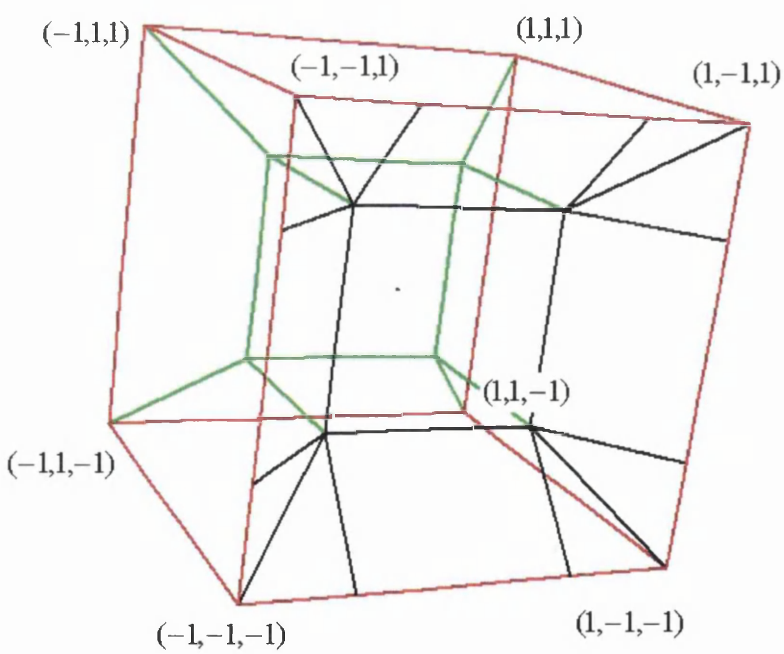

Figure 5.4: Red colour cube makes $C(K(x_0))$ and green colour cube makes $\frac{C(K(x_0))}{1+\lambda}$.

In the Figure 5.4 we describe one side face of $C(K)$ and show different regions mentioned in the above formula of the lower transform of the squared distance function to the set K. Since all the side faces of the cube are symmetric and so we show the symmetry between regions of faces in this figure. Note that the critical point $x_0 := (0,0,0)$ is in the relative interior of $C\big(K(x_0)\big)$ for $K = K(x_0)$ and also note that $\dim C(K) = 3$, that is, $C\big(K(x_0)\big)$ is a three-dimensional polytope. Therefore, from Theorem 3.2.9 the critical point x_0 is a non-degenerate index 3 critical point in the interior of $C\big(K(x_0)\big)$. We note in the previous examples and Theorem 3.2.7 that in the relative interior of convex polygon (a face of $C\big(K(x_0)\big)$ in this example) then there is a non-degenerate index 2 critical point on the face and in relative boundary of convex polygon (relative interior of an edge of this face of $C\big(K(x_0)\big)$ and vertices of K of this face of $C\big(K(x_0)\big)$), then there is a non-degenerate index 1 and index 0 critical points respectively.

The behaviour of solution of initial value problem (5.5.10) of the lower transform when initial condition lies in the relative interior of $C\big(K(x_0)\big)$.

Our aim is to understand the behaviour of solution $\underline{x}(t) := \big(x(t), y(t), z(t)\big)$ of initial value problem (5.5.10) when the initial value $u_0 := \big(x(0), y(0), z(0)\big)$ lies in the relative interior of $C\big(K(x_0)\big)$. In fact, we show that the solution $\underline{x}(t) \to 0$, as $t \to \infty$, that is, the gradient flow converges to the non-degenerate index 3 critical point x_0 as $t \to \infty$. We restrict our investigation to one of the symmertic regions shown in Figure 5.4 for a face of $C\big(K(x_0)\big)$ and infer the behaviour in the other regions by symmetry arguments.

Remark: Note that the explicit formulae for the solution $\underline{x}(t)$ of the first order gradient system of $C^l_\lambda\big((x,y,z),K\big)$ in terms of initial conditions u_0 has been calculated in Appendix B (B.1.7), (B.1.8) and (B.1.9) respectively. Let us first suppose that the initial condition u_0 lies in region (*), that is, $u_0 \in \frac{C(K(x_0))}{1+\lambda}$, then we have

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \to 0, \text{ as } t \to \infty \\ y(t) &= y(0)e^{-2\lambda t} \to 0, \text{ as } t \to \infty \\ z(t) &= z(0)e^{-2\lambda t} \to 0, \text{ as } t \to \infty, \end{aligned}$$

because the term $e^{-2\lambda t} \to 0$ as $t \to \infty$. Furthermore, the gradient in this region,

$$Dh(x, y, z) = -2\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

always points towards x_0, showing that the direction of the gradient of the lower transform is always directly towards the non-degenerate index 3 critical point x_0.

Now we discuss one of the symmetric regions and suppose that the initial condition u_0 lies in region (1) such that $y(0) < \frac{-1}{1+\lambda}$, $|x(0)| \leq \frac{1}{1+\lambda}$ and $|z(0)| \leq \frac{1}{1+\lambda}$. Then from Appendix B (B.1.7),(B.1.8) and (B.1.9)

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \\ y(t) &= \big(y(0) + 1\big)e^{2t} - 1 \\ z(t) &= z(0)e^{-2\lambda t} \end{aligned}$$

If $\frac{-1}{1+\lambda} \leq x(0) < 0$, then $x(t)$ is increasing and thus $x(0) < x(t) < 0$ whenever t is such that $\underline{x}(t)$ lies in region (1). On the other hand, if $0 \leq x(0) < \frac{1}{1+\lambda}$, then $x(t)$ is decreasing and thus $0 < x(t) < x(0)$ whenever t is such that $\underline{x}(t)$ lies in region (1). Analogously $z(t)$ is increasing for $z(0) \in \big(\frac{-1}{1+\lambda}, 0\big)$ and decreasing for $z(0) \in \big(0, \frac{1}{1+\lambda}\big)$. For the behaviour of $y(t)$, we suppose that $-1 < y(0) < \frac{-1}{1+\lambda}$. If $y(0) > -1$, then

$$\dot{y}(t) > 2\big(y(0) + 1\big) > 0.$$

This implies that for $y(0)$ lies inside $C\big(K(x_0)\big)$, the solution $y(t)$ is increasing and hence will intersect the line between region $(*)$ and region (1). Thus the solution $\underline{x}(t)$ can only exit region (1) through the plane with equation of plane given by $y = -\frac{1}{1+\lambda}$, becasue the line through which the solution can exit region (1) is either the plane with equation of plane given by $x = \frac{-1}{1+\lambda}$, $x = \frac{1}{1+\lambda}$, $z = \frac{-1}{1+\lambda}$ or $z = \frac{1}{1+\lambda}$ and this cannot happen because by above arguments for $x(t)$ and $z(t)$ in region (1). So the solution $\underline{x}(t)$ must enter region (*), from which it follows that the solution tends to x_0 as $t \to \infty$, as argued for region

(*). The gradient in region (1),

$$Dh(x,y,z) = \begin{pmatrix} -2\lambda x \\ 2y+2 \\ -2\lambda z \end{pmatrix}.$$

Note that at the boundary between region (*) and (1), that is, on the plane $y = \frac{-1}{1+\lambda}$,

$$\dot{y}(t) = 2y + 2 = \frac{2\lambda}{1+\lambda} > 0.$$

This implies that the gradient of the lower transform points towards $\frac{C(K(x_0))}{1+\lambda}$ and in fact, the gradient flow converges to x_0 as $t \to \infty$.

Remark: Note that as $DC^l_\lambda dist^2\big((x,y,z),K\big)$ is continuous, since $C^l_\lambda dist^2\big((x,y,z),K\big)$ is $C^{1,1}$ and it points directly towards x_0 in region (*). Then it also points directly towards x_0 when it is on the boundary between region (*) and (1).

Suppose that the initial condition u_0 lies in region (2) such that $y(0) < \frac{-1}{1+\lambda}$, $|x(0)| \leq \frac{1}{1+\lambda}$ and $z(0) < \frac{-1}{1+\lambda}$. Then from Appendix B (B.1.7), (B.1.8) and (B.1.9)

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \\ y(t) &= \big(y(0)+1\big)e^{2t} - 1 \\ z(t) &= \big(z(0)+1\big)e^{2t} - 1 \end{aligned}$$

We note that the solutions are similar to the solutions in the regions above. By similar arguments as in region (1), we can argue that the solution will either enter region (*), region (1) or region (5). Let us consider that, for u_0 inside $C\big(K(x_0)\big)$, the solution $\underline{x}(t)$ can exit region (2) through the line of intersection between plane with equation of plane $y = -\frac{1}{1+\lambda}$ and $z = -\frac{1}{1+\lambda}$ and enter region (*) or exit region (2) through plane with equation of plane $z = -\frac{1}{1+\lambda}$ and enter region (1). If the solution enters region (1), then from there it follows solution will enter region (*), as argued for region (1). Once the solution enters region (*), then $\underline{x}(t) \to 0$, as $t \to \infty$ by the arguments in region (*). Analogous to region (1), the gradient $Dh(x,y,z)$ points towards $\frac{C(K(x_0))}{1+\lambda}$ and in fact, the gradient flow converges to x_0 as $t \to \infty$.

Suppose that the initial condition u_0 lies in region (3) such that $x(0) < \frac{-1}{1+\lambda}$, $y(0) < \frac{-1}{1+\lambda}$ and $|z(0)| < \frac{-1}{1+\lambda}$. Then from Appendix B (B.1.7), (B.1.8) and (B.1.9) we have

$$\begin{aligned} x(t) &= \big(x(0)+1\big)e^{2t} - 1 \\ y(t) &= \big(y(0)+1\big)e^{2t} - 1 \\ z(t) &= z(0)e^{-2\lambda t} \end{aligned}$$

The solutions in this region are similar to the solutions in region (2). Therefore, we can see that the solution will either enter region (*), region (1) or region (6). Let us consider for the symmertic reasoning that, for u_0 lies inside $C\big(K(x_0)\big)$, then solution $\underline{x}(t)$ can exit region (3) through the line of intersection between plane $x = -\frac{1}{1+\lambda}$ and $y = -\frac{1}{1+\lambda}$ and enter region (*) or exit region (3) through plane $x = -\frac{1}{1+\lambda}$ and enter region (1). Aanalogous to the arguments in region (2), the solution $\underline{x}(t) \to 0$, as $t \to \infty$, that is, the gradient $Dh(x,y,z)$ points towards $\frac{C(K(x_0))}{1+\lambda}$ and in fact, the gradient flow converges to x_0 as $t \to \infty$.

Suppose that the initial condition u_0 lies in region (4) such that $x(0) < \frac{-1}{1+\lambda}$, $y(0) < \frac{-1}{1+\lambda}$ and $z(0) < \frac{-1}{1+\lambda}$. Then from Appendix B (B.1.7), (B.1.8) and (B.1.9)

$$\begin{aligned} x(t) &= \big(x(0)+1\big)e^{2t} - 1 \\ y(t) &= \big(y(0)+1\big)e^{2t} - 1 \\ z(t) &= \big(z(0)+1\big)e^{2t} - 1. \end{aligned}$$

This obviously shows that for u_0 inside $C\big(K(x_0)\big)$ the solution $\underline{x}(t)$ will enter region (3), which enters region (*) as argued above and hence the solution tends to x_0. The gradient in region (4),

$$Dh(x,y,z) = 2\begin{pmatrix} x+1 \\ y+1 \\ z+1 \end{pmatrix},$$

shows that the gradient points in the direction towards $\frac{C(K(x_0))}{1+\lambda}$ and in fact, the gradient flow converges to x_0 as $t \to \infty$.

Similarly, for the other symmetric regions, it will suffice to argue by symmetry that the solution $\underline{x}(t) \to 0$, as $t \to \infty$. Hence we have shown that when the initial condition lies in the relative interior of $C(K(x_0))$ then x_0 is the non-degenerate index 3 critical point.

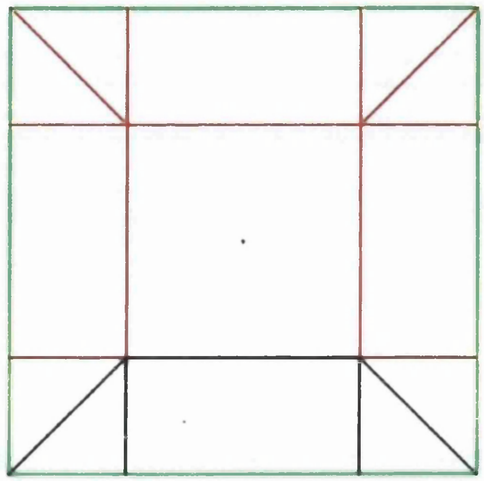

Figure 5.5: The face of $C(K(x_0))$ that passes through points $(-1,-1,1)$, $(-1-,1,-1)$, $(1,-1,1)$, and $(1,-1,-1)$. The convex polygon made by points $(-1,-1,-1)$, $(1,-1,-1)$, $(\frac{-1}{1+\lambda},\frac{-1}{1+\lambda},\frac{-1}{1+\lambda})$ and $(\frac{1}{1+\lambda},\frac{-1}{1+\lambda},\frac{-1}{1+\lambda})$ is symmetric to others polygons.

The behaviour of solution of initial value problem (5.5.10) of the lower transform when initial condition lies in the relative boundary of $C(K(x_0))$.

The relative boundary of $C(K(x_0))$ consists of faces, edges and vertices. From previous Example (5.3.1) and Example (5.4.1), the relative interior of convex polygon, relative interior of edges and vertices are in the stable manifolds of non-degenerate index 2, index 1 and index 0 critical points respectively. Therefore, we will understand the behaviour of $\underline{x}(t)$ for one relative face (convex polygon) of $C(K(x_0))$ and in its relative boundary we consider only the edge such that $-1 < x < 1$ and $y = z = -1$) and a vertex $(-1,-1,-1)$ as shown in the Figure 5.5.

First of all suppose that the initial condition lies in this face of $C(K(x_0))$ such that $y(0) = -1$, $|x(0)| < 1$ and $|z(0)| < 1$. For the relative boundary of $C(K(x_0))$ we choose the **face** (the convex polygon) and by symmetry arguments as for Example (5.4.1) when initial condition lies in the interior of convex polygon, the solution $\underline{x}(t)$ converges to the non-degenerate index 2 critical point, that is, the gradient points towards $(0,-1,0)$.

Therefore, we have

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \to 0 \quad \text{as} \quad t \to \infty \\ y(t) &= (y(0)+1)e^{2t} - 1 = -1 \quad \forall\, t \\ z(t) &= z(0)e^{-2\lambda t} \to 0 \quad \text{as} \quad t \to \infty, \end{aligned}$$

which shows that the solution on the relative boundary of $C\big(K(x_0)\big)$, that is, this face does not converge to the non-degenerate index 3 critical point but does converges to the non-degenerate index 2 critical point which in this case is critical point $(0,-1,0)$. We note that the gradient,

$$Dh(x,y,z) = -2\lambda \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$$

is pointing towards the non-degenerate index 2 critical point $(0,-1,0)$. Similarly, we conclude for other regions of the interior of the convex polygon that the solution will approach to $(0,-1,0)$, that is, the gradient will point towards $(0,-1,0)$.

We consider the **edge** ($-1 < x < 1$ and $y = z = -1$) of the face of $C\big(K(x_0)\big)$ which is contained in region (1) and is the relative boundary of chosen face of $C\big(K(x_0)\big)$. Suppose that the initial condition lies in the relative intrior of this edge, that is, $-1 < x(0) < 1$ and $y(0) = z(0) = -1$. Then,

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \to 0 \quad \text{as} \quad t \to \infty \\ y(t) &= (y(0)+1)e^{2t} - 1 = -1 \quad \forall\, t \\ z(t) &= (z(0)+1)e^{2t} - 1 = -1 \quad \forall\, t, \end{aligned}$$

shows that the solution in the relative boundary of $C\big(K(x_0)\big)$ does not converge to $(0,-1,0)$ but does converge to $(0,-1,-1)$. Note that the gradient,

$$Dh(x,y,z) = \begin{pmatrix} -2\lambda x \\ 0 \\ 0 \end{pmatrix},$$

is pointing towards the non-degenerate index 2 critical point $(0,-1,-1)$. Similarly, for other edges of this face the solution converges to the non-degenerate index 2 critical points in the relative boundary of that edge of face of $C\big(K(x_0)\big)$.

Now consider the **point** $(-1,-1,-1)$ of K and so $x(0)=y(0)=z(0)=-1$. Then,

$$\begin{aligned} x(t) &= (x(0)+1)e^{2t}-1=-1 \quad \forall\, t \\ y(t) &= (y(0)+1)e^{2t}-1=-1 \quad \forall\, t \\ z(t) &= (z(0)+1)e^{2t}-1=-1 \quad \forall\, t, \end{aligned}$$

implies that in the relative boundary of the edge of face of $C\big(K(x_0)\big)$ the solution does not converge to the non-degenerate index 2 critical point $(0,-1,-1)$ but converges to the non-degenerate index 0 critical point $(-1,-1,-1)$. The gradient,

$$Dh(x,y,z)=\begin{pmatrix} 2(x+1) \\ 2(x+1) \\ 2(x+1) \end{pmatrix},$$

points towards $(-1,-1,-1)$. Analogously, for other relative points the solution converges to the corresponding non-degenerate index 0 critical points.

We have shown that the relative interior of the convex polygon (i.e., the relative interior of a face of $C\big(K(x_0)\big)$) is in the stable manifold of the non-degenerate index 2 critical point. Furthermore, we have shown that the relative edges of the chosen face of $C\big(K(x_0)\big)$ (i.e., relative boundary of convex polygon) are in the stable manifold of the non-degenerate index 1 critical points and the vertices of K are in the stable manifold of non-degenerate index 0 critical points. Thus reconstruction gives the union of relative interior of the convex polygons of non-degenerate index 2 and relative boundary i.e., edges of non-degenerate index 1 and vertices of index 0 critical points.

5.6 Degenerate Critical Points

We investigate the behaviour of the solution of first order gradient system of the lower transform $F(x) := \frac{1}{2}C^l_\lambda \mathrm{dist}^2(x,K)$ for the squared distance function to a finite set $K \subset \mathbb{R}^n$

in the neighbourhood of various types of degenerate critical points. The degenerate critical points are defined by 3.2.3 in Chapter 3: if x is a relative boundary point of $C\big(K(x)\big)$, then x is called a degenerate critical point of the lower transform $F(x)$ of the squared distance function to the finite set K. This section is motivated by the fact that degenerate critical points were not studied by Dey [6] for surface reconstruction whereas we can use the degenerate critical points of the lower transform $F(x)$ for surface reconstructions. In fact, we will show that for every initial data in ri $C(K)$ and the part of the relative boundary of $C(K)$ containing the degenerate critical point, the solution of the initial value problem

$$\begin{aligned} \dot{\underline{x}}(t) &= DC_\lambda^l \mathrm{dist}^2\big(\underline{x}(t), K\big) \\ \underline{x}(0) &= u_0 \end{aligned} \tag{5.6.11}$$

is attracted to the degenerate critical point of the lower transform $F(x)$ as $t \to \infty$, that is, the gradient flow of $F(x)$ converges to the degenerate critical point of the lower transform $F(x)$ as $t \to \infty$.

We consider explicit examples of the lower transform $F(\underline{x})$ and illustrate that degenerate critical point x_0 (i.e., degenerate index $(1,2)$, $(2,3)$ or $(1,2,3)$ critical points, see Definition 3.2.4) are in the stable manifolds of degenerate index $(1,2)$, $(2,3)$ or $(1,2,3)$ critical points when initial data u_0 lies inside ri $C\big(K(x_0)\big)$ and the part of the relative boundary of $C\big(K(x_0)\big)$ containing x_0 are in the stable manifolds of degenerate index $(1,2)$, $(2,3)$ or $(1,2,3)$ critical points. In other words, we will show that for every initial value in ri $C\big(K(x_0)\big)$ and the relative boundary of $C\big(K(x_0)\big)$ containing x_0, the solution $\underline{x}(t)$ of the initial value problem (5.6.11) approaches to the degenerate critical point x_0 as $t \to \infty$; that is, the gradient flow of $F(x)$ converges to x_0 as $t \to \infty$. This means that ri $C\big(K(x_0)\big)$ and the relative boundary of $C\big(K(x_0)\big)$ containing x_0 is contained in the stable manifold of the degenerate critical point x_0 and other points are not in the stable manifold of the degenerate critical point x_0.

Example 5.6.1. Let us consider the lower transform $h(x,y) := C_\lambda^l \mathrm{dist}^2\big((x,y), K\big)$ of the squared distance function to a set $K = \{(-1,0),(1,0),(0,1)\}$. For $\lambda > 0$,

$$h(x,y) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2+y^2) & |x| \geq y - \frac{1}{1+\lambda}, y \geq 0 \\ x^2 + (y-1)^2 & |x| \leq y - \frac{1}{1+\lambda} \\ (x+1)^2 + y^2 & x \leq -\frac{1}{1+\lambda}, y \leq -x - \frac{1}{1+\lambda} \\ (x-1)^2 + y^2 & x \geq \frac{1}{1+\lambda}, y \leq x - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{2}\big(x + y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2) & |y-x| < \frac{1}{1+\lambda}, y > -x + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{2}\big(-x + y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2) & |x+y| < \frac{1}{1+\lambda}, x < y - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + y^2 - \lambda x^2 & |x| < \frac{1}{1+\lambda}, y < 0. \end{cases}$$

We denote the regions in above expression by region $(*)$, (1), (2), (3), (4), (5) and region (6) respectively as shown in Figure 5.6. We know from Example (3.3.4) in Chapter 3 that the critical point $x_0 := (0,0)$ is a degenerate index $(1,2)$ critical point of the lower transform $h(x,y)$ of the squared distance function to the set $K = K(x_0)$ and thus x_0 is a boundary point of $C\big(K(x_0)\big)$ where $\dim C\big(K(x_0)\big) = 2$. Therefore, we show that the solution of the initial value problem (5.6.11) converges to the degenerate index $(1,2)$ critical point x_0.

The behaviour of the solution of the initial value problem (5.6.11) when initial condition lies inside $C\big(K(x_0)\big)$.

We investigate that the solution $\underline{x}(t) = \big(x(t), y(t)\big)$ tends to x_0 as $t \to \infty$ when the initial condition $u_0 = \big(x(0), y(0)\big)$ lies in the ri $C\big(K(x_0)\big)$ and the part of the relative boundary of $C\big(K(x_0)\big)$ that contains x_0. To illustrate this, we select one of the symmetric regions and infer the behaviour of the solution in the other regions by symmetry arguments. We classified the regions of $C\big(K(x_0)\big)$ into three regions: region $(*)$, region ω and region ω^* where ω and ω^* are symmetric as shown in Figure 5.6 below. The region ω is defined as a combined region of region (4), part of region (1) and (3). Analogously region ω^* is defined as a combined region of region (5), part of region (1) and (2) as can be seen in the diagram.

Remark: Note that the explicit formulae for the solution $\underline{x}(t)$ of the first order gradient system of $C^l_\lambda dist^2\big((x,y), K\big)$ in terms if initial condition u_0 are calculated in Appendix B

(B.1.3) and (B.1.4).

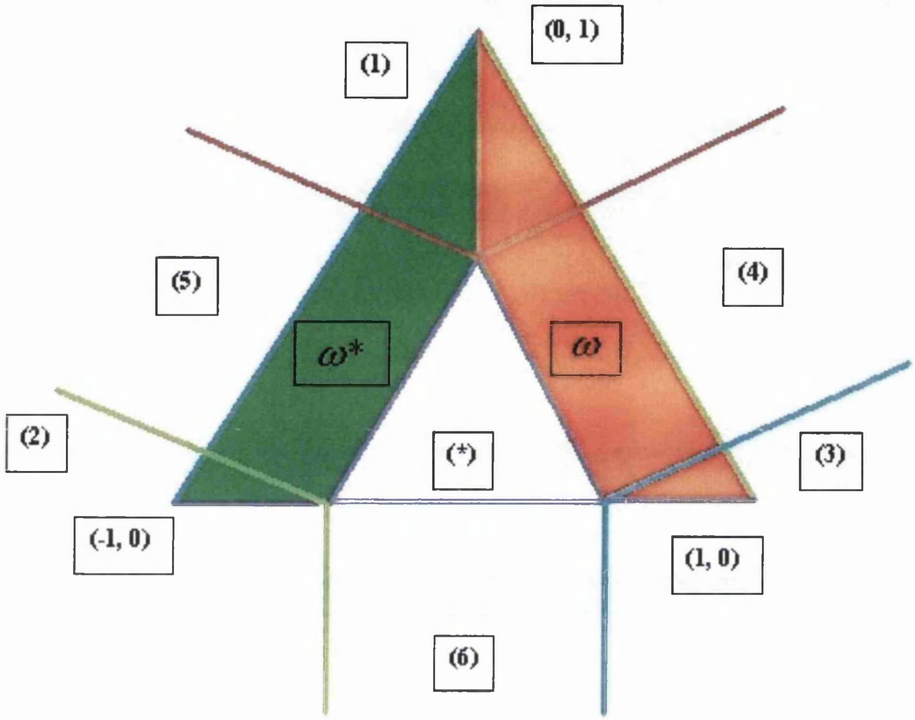

Figure 5.6: The $C(K(x_0))$ and $\frac{C(K(x_0))}{1+\lambda}$, where ω coloured red and ω^* coloured green are symmetric regions of $C(K(x_0))$.

First suppose that the initial condition u_0 lies in region (*), that is, inside $\frac{C(K(x_0))}{1+\lambda}$. Then,

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \to 0 \text{ as } t \to \infty \\ y(t) &= y(0)e^{-2\lambda t} \to 0 \text{ as } t \to \infty, \end{aligned}$$

because the term $e^{-2\lambda t} \to 0$ as $t \to \infty$. Further, note that, the gradient in region (*),

$$Dh(x, y) = -2\lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

always points towards x_0, showing that the direction of the gradient flow of the lower transform is always directly towards the degenerate index $(1, 2)$ critical point x_0.

We now show the behaviour of the solution $\underline{x}(t)$ when the initial condition u_0 lies in region ω and infer the results for ω^* by symmetry arguments. We suppose that the initial condition u_0 lies inside, the part of region (1) $\subset C\big(K(x_0)\big)$ which belongs to ω, the part of region (3) $\subset C\big(K(x_0)\big)$ which belongs to ω, and the part of region (4) $\subset C\big(K(x_0)\big)$ which belongs to ω. Then we show that the solution $\underline{x}(t)$ tends to x_0 as $t \to \infty$ when u_0 lies inside the $C\big(K(x_0)\big)$.

Suppose the initial condition u_0 lies in region (1) such that region (1) $\in \omega$, that is, when $y(0) - x(0) > \frac{1}{1+\lambda}$ and $x(0) > 0$, then the solution is

$$\begin{aligned} x(t) &= x(0)e^{2t} \\ y(t) &= \big(y(0) - 1\big)e^{2t} + 1. \end{aligned}$$

This by substituting value of $x(t)$ gives us the following equation of straight line

$$y(t) = \Big(\frac{y(0) - 1}{x(0)}\Big)x(t) + 1, \tag{5.6.12}$$

which passes thought point $(0, 1)$ and thus for given initial condition it must enter region (4) through line $y = x + \frac{1}{1+\lambda}$. Then we will show next that the solution tends to x_0 when initial condition lies inside region (4) $\in \omega$ as follows.

Suppose the initial condition u_0 lies in region (4) $\in \omega$, that is, when $|x(0)-y(0)| < \frac{1}{1+\lambda}$ and $\frac{1}{1+\lambda} < y(0) + x(0) < 1$. The differential equations in this region are

$$\begin{aligned} \dot{x}(t) &= (1 - \lambda)x + (1 + \lambda)y - 1 \\ \dot{y}(t) &= (1 + \lambda)x + (1 - \lambda)y - 1, \end{aligned}$$

and so,

$$\begin{aligned} \dot{x}(t) + \dot{y}(t) &= 2(x + y) - 2 \\ \dot{x}(t) - \dot{y}(t) &= -2\lambda(x - y). \end{aligned}$$

Let us suppose that $\alpha(t) := y(t) - x(t)$, and $\beta(t) := y(t) + x(t)$, then $\dot{\alpha}(t) = \dot{y}(t) - \dot{x}(t)$ and $\dot{\beta}(t) = \dot{y}(t) + \dot{x}(t)$. Hence,

$$\dot{\alpha}(t) = -2\lambda\alpha(t), \quad \text{and} \quad \dot{\beta}(t) = 2\beta - 2.$$

So, $\alpha(t) = \alpha(0)e^{-2\lambda t}$ shows that the gradient along the line $y - x = \frac{1}{1+\lambda}$ and $y - x = -\frac{1}{1+\lambda}$ is always towards region (4) $\in \omega$. Thus the solution can only exit region (4) through boundary line between region (*) and region (4). Now $\beta(t) = (\beta(0) - 1)e^{2t} + 1$ shows that the gradient along the line $y + x = \frac{1}{1+\lambda}$ is always directly towards region (*) because $\beta(0) - 1 < 0$. Once the solution reaches region (*), then by arguments of region (*) it tends to degenerate index (1,2) critical point x_0. Analogously, we can inferred for the symmetric region ω^* and hence we conclude that the relative interior of $C\big(K(x_0)\big)$ and relative boundary of $C\big(K(x_0)\big)$ containing x_0 is contained in the stable manifold of degenerate index (1,2) critical point.

Example 5.6.2. We consider the lower transform $h(x, y, z) := C_\lambda^l \operatorname{dist}^2\big((x, y, z), K\big)$ of the squared distance function to a set $K = \Big\{\big(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big), \big(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big), (0, 1, 0), (0, 0, 1)\Big\}$. This is an extension of Example (4.2.3) into $\mathbb{R}^3$ by adding point $(0, 0, 1)$ to it. In this example the critical point lies on one face (the plane z = 0) of $C\big(K(x_0)\big)$ where $\dim C\big(K(x_0)\big) = 3$ and $K = K(x_0)$, that is, Example (4.2.3) with $z = 0$. From the formula of $h(x, y, z)$ we choose region (1) and (2) which are directly adjacent to the critical point $x_0 = (0, 0, 0)$. The formulae of the lower transform $h(x, y, z)$ for region (1) and (2) are

$$h(x, y, z) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2 + y^2 + z^2) & (1) \\ \frac{\lambda}{1+\lambda} - \lambda(x^2 + y^2) + z^2 & (2). \end{cases}$$

The partial first order differential equations with respect to x, y and z in region (1),

$$\begin{aligned} \dot{x}(t) &= -2\lambda x & \Longrightarrow\ & x(t) = x(0)e^{-2\lambda t} \\ \dot{y}(t) &= -2\lambda y & \Longrightarrow\ & y(t) = y(0)e^{-2\lambda t} \\ \dot{z}(t) &= -2\lambda z, & \Longrightarrow\ & z(t) = z(0)e^{-2\lambda t}, \end{aligned}$$

and in region (2) only $\dot{z}(t) = 2z$, is different. Note that from Theorem 3.2.9 in Chapter 3 the critical point x_0 in this example is a degenerate index $(2, 3)$ critical point which is also illustrated as a prototype Example (3.3.5) of Morse indices. This implies that x_0 is a relative boundary point of $C\big(K(x_0)\big)$ and so we show that when the initial condition u_0 is in the relative interior or in the relative boundary of $C\big(K(x_0)\big)$ containing x_0, then the solution of the initial value problem (5.6.11) converges to degenerate index $(2,3)$ critical point x_0.

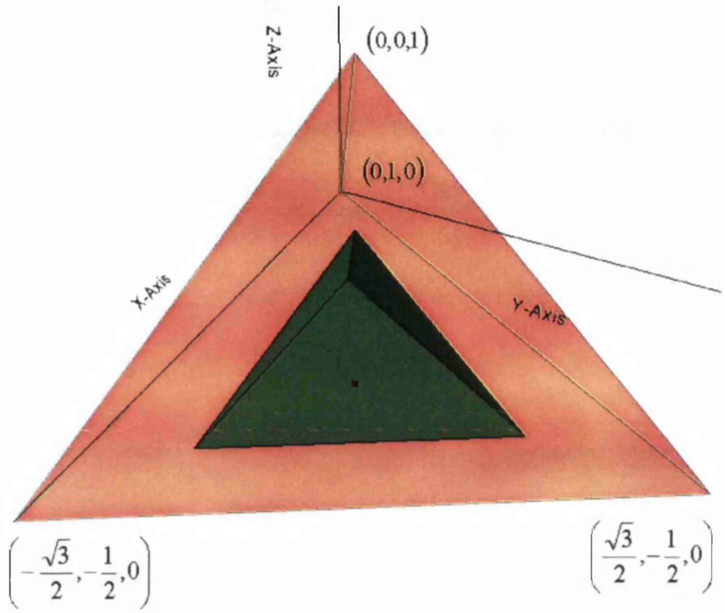

Figure 5.7: The Tetrahedron in this diagram is plotting of $C\big(K(x_0)\big)$ and $\frac{C(K(x_0))}{1+\lambda}$ in $\mathbb{R}^3$ with the critical point x_0 denoted by a dot lies on a face (plane z = 0) of the Tetrahedron. The region (1) denotes the smaller Tetrahedron inside and region (2) denotes the adjacent outside of the face having critical point on it.

We will show that the solution $\underline{x}(t) = \big(x(t), y(t), z(t)\big)$ of the initial value problem (5.6.11) tends to x_0 as $t \to \infty$ when the initial condition $u_0 = \big(x(0), y(0), z(0)\big)$ lies in region (1) and it does not tends to x_0 when the initial condition u_0 lies in region (2). Therefore, suppose that the initial condition u_0 lies in region (1), that is, inside $\frac{C(K(x_0))}{1+\lambda}$. Then,

$$\begin{aligned} x(t) &= x(0)e^{-2\lambda t} \to 0 \text{ as } t \to \infty \\ y(t) &= y(0)e^{-2\lambda t} \to 0 \text{ as } t \to \infty \\ z(t) &= z(0)e^{-2\lambda t} \to 0 \text{ as } t \to \infty, \end{aligned}$$

clearly shows that the solution $\underline{x}(t) \to x_0$ as $t \to \infty$, because the term $e^{-2\lambda t} \to 0$ as

$t \to \infty$. Further, note that, the gradient in region (1),

$$Dh(x, y, z) = -2\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

always points towards x_0, showing that the direction of the gradient flow of the lower transform is always directly towards the degenerate index $(2, 3)$ critical point x_0.

To show that for initial condition u_0 lies inside region (2) it is sufficient to show that for $z(0) < 0$, the solution $z(t) \to -\infty$ as $t \to \infty$. Therefore, suppose that the initial condition u_0 lies inside region (2). Then,

$$z(t) = z(0)e^{2t} \to -\infty \text{ as } t \to \infty,$$

shows that the solution $\underline{x}(t)$ does not converge to x_0 as $t \to \infty$. Hence we conclude that the relative interior of $C\big(K(x_0)\big)$ and the relative boundary of $C\big(K(x_0)\big)$ containing x_0 is contained in the stable manifold of degenerate index (2,3) critical point x_0 and the outside of $C\big(K(x_0)\big)$ is not in the stable manifold of degenerate index (2,3) critical point x_0.

Example 5.6.3. We consider the lower transform $h(x, y, z) := C^l_\lambda \mathrm{dist}^2\big((x, y, z), K\big)$ of the squared distance function to a set $K = \Big\{\big(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\big), \big(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\big), (0, 0, 1), (0, 0, -1)\Big\}$. In this example the critical point $x_0 = (0, 0, 0)$ lies on one edge (i.e., the line of intersection of four planes, $\sqrt{3}x + y = 0$, $-\sqrt{3}x + y = 0$, $-x + \sqrt{3}y = 0$,$x + \sqrt{3}y = 0$) of $C\big(K(x_0)\big)$ where $\dim C\big(K(x_0)\big) = 3$ and $K = K(x_0)$. From the explicit formula for the lower transform $h(x, y, z)$ in Formula (4.2.5) we choose region (1), (2), (3) and (4) which are directly adjacent to the edge containing the critical point x_0. The formulae of the lower transform $h(x, y, z)$ for these regions is given by

$$h = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2 + y^2 + z^2) & (1) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}(-x + \sqrt{3}y)^2 - \lambda(x^2 + y^2 + z^2) & (2) \\ \frac{\lambda}{1+\lambda} + \frac{(1+\lambda)}{4}(x + \sqrt{3}y)^2 - \lambda(x^2 + y^2 + z^2) & (3) \\ \frac{\lambda}{1+\lambda} + (1 + \lambda)(x^2 + y^2) - \lambda(x^2 + y^2 + z^2) & (4) \end{cases}$$

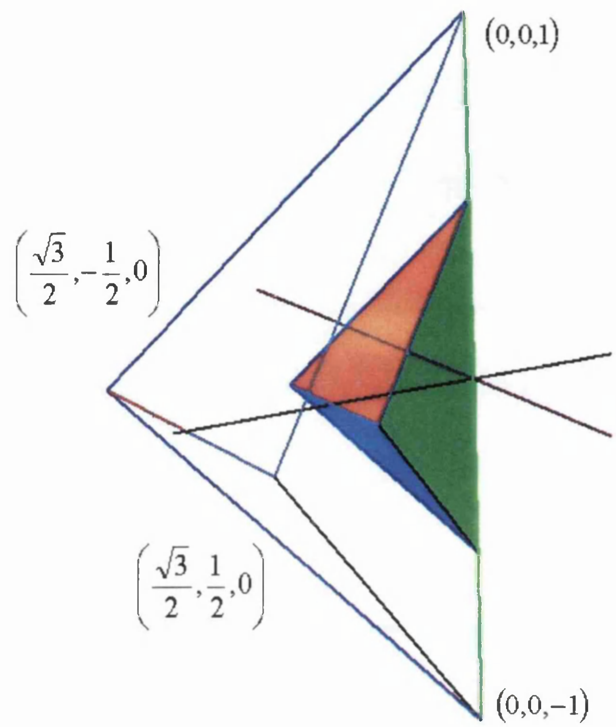

Figure 5.8: The Tetrahedron is $C(K(x_0))$ in $\mathbb{R}^3$ with the critical point x_0 on the edge, that is, on the line of intersection of plane $\sqrt{3}x + y = 0$, $-\sqrt{3}x + y = 0$, $-x + \sqrt{3}y = 0$ and $x + \sqrt{3}y = 0$. The region (1) denotes the smaller Tetrahedron, region (2) is between the green colour plane and perpendicular plane $\sqrt{3}x + y = 0$ to it, region (3) is between $\sqrt{3}x + y = 0$ and $-\sqrt{3}x + y = 0$, and region (4) is beteen plane $x + \sqrt{3}y = 0$ and $-\sqrt{3}x + y = 0$.

In the Figure 5.8 we plot the $C(K(x_0))$ and $\frac{C(K(x_0))}{1+\lambda}$ in $\mathbb{R}^3$. We investigate the behaviour of the solution $\underline{x}(t) = (x(t), y(t), z(t))$ of the initial value problem (5.6.11) of the lower transform of the squared distance function to the set K. From Theorem 3.2.9 in Chapter 3 the critical point x_0 of this example is a degenerate index $(1, 2, 3)$ critical point and also illustated as prototype Example (3.3.6) of Morse indices. Since x_0 is a relative boundary point of $C(K(x_0))$ and so we will show that the solution $\underline{x}(t)$ converge to x_0 when the initial condition $u_0 = (x(0), y(0), z(0))$ lies in ri $C(K(x_0))$ and in the relative boundary of $C(K(x_0))$ containing x_0. The partial first order differential equation with respect to z is same for all four regions, so the partial differential equation is $\dot{z}(t) = -2\lambda z$, and its solution is $z(t) = z(0)e^{-2\lambda t}$.

In region (1) the partial first order differential equations with respect to x and y are

$$\begin{aligned}\dot{x}(t) &= -2\lambda x, \text{ and its soluton is } x(t) = x(0)e^{-2\lambda t}, \\ \dot{y}(t) &= -2\lambda y, \text{ and its soluton is } y(t) = y(0)e^{-2\lambda t}.\end{aligned}$$

This shows that when the initial condtion u_0 are inside region (1), then the solution $\underline{x}(t) \to x_0$ as $t \to \infty$, because the term $e^{-2\lambda t} \to 0$ as $t \to \infty$.

In region (2) the partial first order differential equations with respect to x and y, and their solutions are

$$\begin{aligned}\dot{x}(t) &= \frac{1-3\lambda}{2}x - \frac{\sqrt{3}(1+\lambda)}{2}y \\ x(t) &= \frac{1}{4}\big(x(0) - \sqrt{3}y(0)\big)e^{2t} + \frac{\sqrt{3}}{4}\big(\sqrt{3}x(0) + y(0)\big)e^{-2\lambda t}, \\ \dot{y}(t) &= -\frac{\sqrt{3}(1+\lambda)}{2}x + \frac{3-\lambda}{2}y \\ y(t) &= -\frac{\sqrt{3}}{4}\big(x(0) - \sqrt{3}y(0)\big)e^{2t} + \frac{1}{4}\big(\sqrt{3}x(0) + y(0)\big)e^{-2\lambda t}\end{aligned}$$

If we suppose $x(0) - \sqrt{3}y(0) = 0 \implies x(0) = \sqrt{3}y(0)$, then solution $\underline{x}(t) \to x_0$ as $t \to \infty$.

In region (3) the partial first order differential equations with respect to x and y, and their solutions are

$$\begin{aligned}\dot{x}(t) &= \frac{1-3\lambda}{2}x + \frac{\sqrt{3}(1+\lambda)}{2}y \\ x(t) &= \frac{1}{4}\big(x(0) + \sqrt{3}y(0)\big)e^{2t} + \frac{\sqrt{3}}{4}\big(\sqrt{3}x(0) - y(0)\big)e^{-2\lambda t}, \\ \dot{y}(t) &= \frac{\sqrt{3}(1+\lambda)}{2}x + \frac{3-\lambda}{2}y \\ y(t) &= \frac{\sqrt{3}}{4}\big(x(0) + \sqrt{3}y(0)\big)e^{2t} - \frac{1}{4}\big(\sqrt{3}x(0) - y(0)\big)e^{-2\lambda t}\end{aligned}$$

If $x(0) + \sqrt{3}y(0) = 0 \implies x(0) = -\sqrt{3}y(0)$, then solution $\underline{x}(t) \to x_0$ as $t \to \infty$.

In region (4) the partial first order differential equations with respect to x and y are

$$\begin{aligned}\dot{x}(t) &= 2x, \text{ and its soluton is } x(t) = x(0)e^{2t} \\ \dot{y}(t) &= 2y, \text{ and its soluton is } y(t) = y(0)e^{2t}\end{aligned}$$

In this region the solution $\underline{x}(t) = x_0$ only when $x(0) = y(0)$. Hence, we note that in region (1), the solution $\underline{x}(t)$ tends to x_0 for all t and in other region it tends to x_0 for restricted initial condition u_0.

In this chapter we looked at the local (i.e., on $C(K(x_0)))$ and global (i.e., on $C(K))$ reconstructions for finite sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, where x_0 is the critical point of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to the finite set K. We have shown for some explicitly calculated examples of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ that the interior of $C\big(K(x_0)\big)$ is in the stable manifolds of non-degenerate index 2 critical points as well as degenerate critical points.

Chapter 6

Applications to ordinary differential equations and early universe model

In this chapter we investigate the long-time dynamics of the second order gradient systems governed by the lower transform $\frac{1}{2}C_{\lambda}^{l}\mathrm{dist}^2(x, K)$ of the squared distance functions to a specific finite set $K = \{-1, 1\} \subset \mathbb{R}$ and $K = \{(-1, 0), (1, 0)\} \subset \mathbb{R}^2$. We prove the existence of unique global solution of the initial value problem of the lower transform $C_{\lambda}^{l}\mathrm{dist}^2(x, K)$ for general closed sets $K \subset \mathbb{R}^n$. The existence of unique global solution is one of the motivations to investigate the dynamics of the second order gradient system of the lower transform $C_{\lambda}^{l}\mathrm{dist}^2(x, K)$ as compared to the Brenier dynamical system [5] governed by half of the squared distance functon $\frac{1}{2}\mathrm{dist}^2(x, K)$. The lower transform $C_{\lambda}^{l}\mathrm{dist}^2(x, K)$ gives a tight smoothing and therefore smoothes of the singularities yet allow solutions of the second order gradient system of the lower transform to keep the original dynamics properties. In the case of Brenier model, the solution is no longer unique after reaching singularities in finite time, whereas in our model the lower transform $C_{\lambda}^{l}\mathrm{dist}^2(x, K)$ automatically take cares of the singularities because of global existence of a unique solution. Therefore, the global existence of a unique solution for gradient system of $C_{\lambda}^{l}\mathrm{dist}^2(x, K)$ is a big benefit over the original Brenier model [5].

In Section 1, we show global existence and uniqueness of the solution for the dynamical system governed by the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to a closed set $K \subset \mathbb{R}^n$. Then in Section 2, we study the Monge-Ampere gravitaion model [5] of Brenier using the potential $\Phi(x) = \frac{1}{2}\mathrm{dist}^2(x,K)$ for the set $K = \{-1,1\}$ and compare its dynamical behaviour with that of the second order gradient system using the lower transform $\frac{1}{2}C^l_\lambda \mathrm{dist}^2(x,K)$ for the squared distance functions to K. We use the solutions of second order gradient systems of the lower transform $\frac{1}{2}C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to set $K = \{-1,1\}$ and set $K = \{(-1,0),(1,0)\}$, to investigate the effects of smoothing on the longtime dynamics of the systems for a large fixed $\lambda > 0$. These second order gradient systems and explicit formulae for their solutions of the examples based on the lower transforms will be calculated in Chapter 7 in detail. The investigation shows the oscillating behaviour of solutions when the initial data $\big(x(0), \dot{x}(0)\big)$ are restricted.

In Section 3, we study qualitative behvaiour of the solutions of second order gradient systems of the lower transform $\frac{1}{2}C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to set $K = \{-1,1\}$ based on a conservation law, that is,

$$\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda(x(t)) = \frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)), \tag{6.0.1}$$

where $F_\lambda(x(t)) = \frac{1}{2}C^l_\lambda \mathrm{dist}^2(x,K)$. Then we establish some convergence results for the solutions when $\lambda \to \infty$. Finally, we investigate the longtime behaviour of the solutions of first order gradient system of the lower transform $\frac{1}{2}C^l_\lambda \mathrm{dist}^2(x,K)$ to set $K = \{-1,1\}$.

6.1 Existence of Unique Global Solution

Let us consider the initial value problem (IVP) with $f(x) := \frac{1}{2}DC^l_\lambda \mathrm{dist}^2\Big(x,K\Big)$, where K is a finite subset of $\mathbb{R}^n$. Then

$$\begin{aligned} \ddot{x}(t) &= f\big(x(t)\big) \\ x(0) &= u_0 \\ \dot{x}(0) &= w_0, \qquad\qquad \text{where } u_0, w_0 \in \mathbb{R}^n. \end{aligned} \tag{6.1.2}$$

We want to show that this initial value problem (IVP) has a unique global solution. For this we first convert this second order system of differential equations (IVP) into a system of first order differential equations. Therefore, let $w(t) = \dot{x}(t)$ and so

$$\dot{w}(t) = \ddot{x}(t) = f\big(x(t)\big)$$

with $x(0) = u_0$ and $w(0) = w_0$. Then, the system of second order differential equations (6.1.2) can be re-written as the system of first order differential equations

$$\begin{bmatrix} \dot{w}(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) \\ w(t) \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ w_0 \end{bmatrix}$$

We have to show that there exists a unique global solution for this first order initial value problem and hence we need to show global Lipschitz continuity. Note that for a Lipschitz continuous function (see Definition 2.2.8), Theorem 2.2.17 states that the gradient of the lower transform for the squared distance function to the finite set K is globally Lipschitz continuous and thus we have the following Lemma.

Lemma 6.1.1. *Suppose K is a non-empty non-convex and closed subset of $\mathbb{R}^n$. Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be a function defined by*

$$F(x, w) = \begin{bmatrix} f(x) \\ w \end{bmatrix}$$

where $(x, w) \in \mathbb{R}^n \times \mathbb{R}^n$. Then F is globally Lipschitz continuous.

Proof. We prove that $F(x, w)$ is globally lipschitz continuous, that is, there exist $M > 0$ such that for all $x, \tilde{x}, w, \tilde{w} \in \mathbb{R}^n$,

$$\|F(x, w) - F(\tilde{x}, \tilde{w})\| \leq M\sqrt{\|x - \tilde{x}\|^2 + \|w - \tilde{w})\|^2}$$

From Theorem [20, Theorem 3.1] we know that $f(x(t))$ is globally Lipschitz continuous, That is, there exists $L \in (0, 8 + 10\lambda]$ such that for all $x, \tilde{x} \in \mathbb{R}^n$,

$$\|f(x) - f(\tilde{x})\| \leq L\|x - \tilde{x}\|$$

Since

$$\|F(x,w) - F(\tilde{x},\tilde{w})\| \le \|f(x) - f(\tilde{x})\| + \|w - \tilde{w}\|,$$

from the global Lipschitz continuity of $f(x(t))$, we obtain

$$\begin{aligned} \|F(x,w) - F(\tilde{x},\tilde{w})\| &\le L\|x-\tilde{x}\| + \|w-\tilde{w}\| \\ &\le \sqrt{L^2+1}\sqrt{\|x-\tilde{x}\|^2 + \|w-\tilde{w}\|^2} \\ &\le M\sqrt{\|x-\tilde{x}\|^2 + \|w-\tilde{w}\|^2} \end{aligned}$$

This implies that $F(x,w)$ is globally Lipschitz continuous. □

Since it has been shown that $F(x,w)$ is globally Lipschitz continuous, we state the following theorem from [10] about the existence of unique global solution of globally Lipschitz continuous functions.

Theorem 6.1.2. *[10, Section 3.2] Suppose that $g(t,0)$ is locally bounded as a function of t, and that there exists $\beta > 0$ such that $\|g(t,u) - g(t,v)\| \le \beta\|u-v\|$ for all $t \in (-\infty,\infty)$ and all points $u,v \in X$ (where X is n-dimensional Euclidean space). Then there is a unique global solution $u \in C\Big((-\infty,\infty),X\Big)$ of the initial value problem*

$$\dot{u}(t) = g(t,u(t)), \quad u(0) = u_0 \in X.$$

Hence the global Lipschitz continuity of $F(x,w)$ and Theorem 6.1.2 implies the unique global solution of the initial value problem (6.1.2).

6.2 Smoothing effects on the dynamics of gradient systems and Monge-Ampere gravitation model

In this section we investigate the longtime dynamics of second order smoothed gradient systems governed by the lower transform for the squared distance function to a finite set $K \subset \mathbb{R}^n$. Our claim is to show by using examples that the solution of second order gradient system of the lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to some specific sets K will either oscillate within $C(K)$ or approach infinity. We compare the behaviour of solutions for both second order gradient system of the lower transform for the squared distance function to the finite set K and second order gradient system of Brenier [5, dynamical equation 2.15] (Monge-Ampere gravitation model), which is motivation of this section.

We first elaborate the framework for Monge-Ampere gravitation model and then see how does the solution of dynamical system used for this model is behaving as compare to our model. The general dynamical system of Brenier is governed by the potential Φ:

$$\frac{d^2x}{dt^2} = D\Phi(x), \tag{6.2.3}$$

where D denotes the gradient operator and Φ defined as half of the squared distance to a bounded set K, that is,

$$\Phi(x) := \inf\left\{\frac{|x-s|^2}{2};\ s \in K\right\} = \frac{1}{2}\mathrm{dist}^2(x, K).$$

In this model, $\pi(x)$ is supposed to be the unique closest point of set K to x if $x \neq M_K$, the medial axis of set K, and so $x(0) = u_0$ has a unique solution $\pi(x_0) = \pi_0$ in the set K. Thus an explicit solution of (6.2.3) constructed by Brenier [5] is given by

$$x(t) = \pi_0 + (x_0 - \pi_0)e^t. \tag{6.2.4}$$

The global existence of unique solution of the system (6.2.3) using $C^l_\lambda \mathrm{dist}^2(x, K)$ instead of $\Phi(x)$ (see Theorem 6.1.2 and [20, Theorem 3.1]) changes the behaviour of solution. Using the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$, the behaviour of the solution $x(t)$ of the system is more clear even on the singularities of Φ, because the lower transform for the squared distance functions to finite sets gives a way of continuing the solution. On the other hand, the Brenier dynamical system 6.2.3 is a non-smooth dynamical system since Φ is not differentiable at points on M_K. Therefore, when the solution approaches M_K in finite time it is not obvious what happens on the singularities and thus the existence and behaviour of the solutions in Brenier model is not clear in general. This motivates us to use this specific method of smoothing to study the non-smooth dynamical system (6.2.3) of Brenier and investigate the behaviour of the solution $x(t)$.

To understand the behaviour of the solution $x(t)$ of the second order gradient system (6.2.3) governed instead by the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of squared distance function to the finite set K, we modify the second order gradient system (6.2.3) as follows

$$\frac{d^2x}{dt^2} = \frac{1}{2} D C^l_\lambda \mathrm{dist}^2(x, K) \tag{6.2.5}$$

where D denotes the gradient operator and K is a finite subset of $\mathbb{R}^n$. We know from Lemma 3.1.2 in Chapter 3 that,

$$C^l_\lambda \mathrm{dist}^2(x, K(0)) = \frac{\lambda}{1+\lambda} r^2 + (1+\lambda)\mathrm{dist}^2\Big(x, \frac{C(K(0))}{1+\lambda}\Big) - \lambda|x|^2.$$

Then the second order gradient system (6.2.5) can be written as

$$\ddot{x}(t) = x(t) - (1+\lambda)x_\lambda(t),$$

where $x_\lambda(t) \in \frac{C(K(0))}{1+\lambda}$ is the projection of $x(t)$ onto $\frac{C(K(0))}{1+\lambda}$. If we suppose, for the moment, that $(1+\lambda)x_\lambda(t)$ is the unique closest point of the set K to $x(t)$, that is, $\pi(x) = (1+\lambda)x_\lambda$ then compare to the formula (6.2.4) and we still have Brenier's the solution

$$x(t) = \pi_0 + (x_0 - \pi_0)e^t$$

will move from its initial position in the direction opposite to its closest point π_0 in the set K. When $\pi(x)$ and $(1+\lambda)x_\lambda$ are not the same, then for certain restrictions on the initial condition $x(0)$ and $\dot{x}(0)$, for (6.2.5), for small $t > 0$, we calculate $A = \frac{1}{2}\big(\dot{x}(0) + x(0) - (1+\lambda)x_\lambda(0)\big)$ and $B = \frac{1}{2}\big(-\dot{x}(0) + x(0) - (1+\lambda)x_\lambda(0)\big)$. Therefore, the solution locally in the form when $x_\lambda(t) = x_\lambda(0)$ is,

$$x(t) = (1+\lambda)x_\lambda(0) + \frac{1}{2}\big(\dot{x}(0) + x(0) - (1+\lambda)x_\lambda(0)\big)e^t + \frac{1}{2}\big(-\dot{x}(0) + x(0) - (1+\lambda)x_\lambda(0)\big)e^{-t},$$

moves from its initial condition in the direction opposite to its closest point in the set K under certain restrictions on the initial condition $x(0)$ and $\dot{x}(0)$.

In the following, we study a one-dimensional example. Let $K = \{-1, 1\}$, and we investigate the behaviour of the solution $x(t)$ and its derivative $\dot{x}(t)$ of the second order gradient system of $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to the finite set K. Let

$F_\lambda(x) = \frac{1}{2}C_\lambda^l \mathrm{dist}^2(x, K)$. If we suppose $-1 < x(0) < 1$ and $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda\big(x(0)\big) < 0$, then the solution $x(t)$ oscillates between -1 and 1. We also establish conservation law

$$\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda\big(x(t)\big) = \frac{1}{2}|\dot{x}(0)|^2 - F_\lambda\big(x(0)\big), \tag{6.2.6}$$

which determines the qualitative behaviour of solutions.

First of all we show that if we take the initial values $x(0)$ between $-\frac{1}{1+\lambda}$ and $\frac{1}{1+\lambda}$, then the solution $x(t)$ moves away from its closest point 1 of set K and continue to moves with initial velocity $\dot{x}(0)$ till another point -1 of set K is more closer than 1 to $x(t)$. We compute the condition on the initial velocity such that the solution $x(t)$ oscillates between -1 and 1. Secondly, we suppose the initial condition $x(0) \in \big(\frac{1}{1+\lambda}, 1\big)$. Then for some time $t_0 \geq 0$ such that $0 \leq t_0$ the solution $x(t)$ moves away from 1 and reaches $x(t_0) = \frac{1}{1+\lambda}$ at time t_0 and move to the modified region. The solution go through the modified region and for some time $\hat{t} \geq 0$ such that $t_0 < \hat{t}$ the solution $x(t)$ reaches $x(\hat{t}) = -\frac{1}{1+\lambda}$. Then, depending on the initial data, the solution $x(t)$ can possibly get to $-\infty$ or turns around. In the following example, we consider the case when $x(t)$ turns around and move back towards the modified region.

Example 6.2.1. The explicit formula for the lower transform $F_\lambda(x) = \frac{1}{2}C_\lambda^l \mathrm{dist}^2(x, K)$ of the squared distance function to set $K = \{-1, 1\}$ from Example (4.2.1) in Chapter 4 for $\lambda > 0$ is given by

$$F_\lambda(x) = \begin{cases} |x-1|^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda|x|^2 & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ |x+1|^2 & x < -\frac{1}{1+\lambda} \end{cases}. \tag{6.2.7}$$

The second order gradient system of $F_\lambda(x)$ is

$$\ddot{x}(t) = \begin{cases} x-1 & x > \frac{1}{1+\lambda} \\ -\lambda x & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ x+1 & x < -\frac{1}{1+\lambda} \end{cases}, \tag{6.2.8}$$

and the explicit formula for the solution of this second order gradient system is given by

$$x(t) = \begin{cases} A_1 e^t + A_2 e^{-t} + 1 & x > \frac{1}{1+\lambda} \\ B_1 \cos(\sqrt{\lambda}t) + B_2 \sin(\sqrt{\lambda}t) & -\frac{1}{1+\lambda} \le x \le \frac{1}{1+\lambda} \\ C_1 e^t + C_2 e^{-t} - 1 & x < -\frac{1}{1+\lambda} \end{cases}.$$

The formula for solution in terms of the initial condition $x(0)$ and $\dot{x}(0)$ is as follows.

$$x(t) = \begin{cases} \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^t + \frac{1}{2}\big(-\dot{x}(0) + x(0) - 1\big)e^{-t} + 1 & x(0) > \frac{1}{1+\lambda} \\ x(0)\cos(\sqrt{\lambda}t) + \frac{1}{\sqrt{\lambda}}\dot{x}(0)\sin(\sqrt{\lambda}t) & -\frac{1}{1+\lambda} \le x(0) \le \frac{1}{1+\lambda} \\ \frac{1}{2}\big(\dot{x}(0) + x(0) + 1\big)e^t + \frac{1}{2}\big(-\dot{x}(0) + x(0) + 1\big)e^{-t} - 1 & x(0) < -\frac{1}{1+\lambda} \end{cases} \tag{6.2.9}$$

Case (i): If $-1 < x(0) < 1$ and assume $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) < 0$, then the solution $x(t)$ oscillates between 1 and -1.

First we investigate the behaviour of the solution when the initial data inside the modified region. Let us suppose that $x(0) \in \big(-\frac{1}{1+\lambda}, \frac{1}{1+\lambda}\big)$. Then, the solution is given by

$$x(t) = x(0)\cos(\sqrt{\lambda}t) + \frac{\dot{x}(0)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}t). \tag{6.2.10}$$

Let $E := \sqrt{\big(x(0)\big)^2 + \big(\frac{\dot{x}(0)}{\sqrt{\lambda}}\big)^2}$, then the formula for the solution $x(t)$ from (6.2.10) can be written as

$$x(t) = E\Big(\frac{x(0)}{E}\cos(\sqrt{\lambda}t) + \frac{\dot{x}(0)}{\sqrt{\lambda}E}\sin(\sqrt{\lambda}t)\Big).$$

Now we set $\sin(\phi) := \frac{x(0)}{E}$ and $\cos(\phi) := \frac{\dot{x}(0)}{\sqrt{\lambda}E}$, then the trignometric formula gives that

$$x(t) = E\sin(\phi + \sqrt{\lambda}t).$$

Since $-1 \le \sin(\phi + \sqrt{\lambda}t) \le 1$ and $-\frac{1}{1+\lambda} < x(0) < \frac{1}{1+\lambda}$. Therefore, $-\frac{1}{1+\lambda} < x(t) < \frac{1}{1+\lambda}$ if and only if $E < \frac{1}{1+\lambda}$ which implies that $|\dot{x}(0)| < \sqrt{\frac{\lambda}{(1+\lambda)^2} - \lambda\big(x(0)\big)^2}$. However, this cannot happen for all $t > 0$ as the conservation law (6.2.6)

$$\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda(x(t)) = \frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)),$$

will force $\big(x(t), \dot{x}(t)\big)$ to stay on a closed curve. Therefore, $E > \frac{1}{1+\lambda}$ and the solution will enter other regions.

We try to understand the behaviour of the solution when the initial data is between $\frac{1}{1+\lambda}$ and 1. Let us suppose that $\frac{1}{1+\lambda} < x(0) < 1$ and assume $\alpha := \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big) < 0$ and $\beta := \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big) < 0$. Then is this case from Formula (6.2.8),

$$\ddot{x}(t) = x(t) - 1 < 0,$$

and therefore,

$$\dot{x}(t) < \dot{x}(0) < 0.$$

This shows that the solution $x(t)$ is not only decreasing but decreasing at least as quickly as it decreases at 0 and thus will eventually reach $\frac{1}{1+\lambda}$ at some time t_0 given by

$$t_0 = \ln \frac{-\frac{\lambda}{1+\lambda} - \sqrt{\frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta}}{2\alpha},$$

(see Appendix A.1 for details of time t_0).

Now the solution contacts the boundary of the modified region at time t_0, so the solution $x(t)$ from Formula (6.2.9) for $x(t_0) = \frac{1}{1+\lambda}$ can be written as

$$B_1 \cos(\sqrt{\lambda}t) + B_2 \sin(\sqrt{\lambda}t) = \alpha e^{t_0} + \beta e^{-t_0} + 1,$$

which gives values of constants B_1 and B_2 (see Appendix C for calculation of B_1 and B_2). The formula for solution $x(t)$ can also be written as

$$\begin{aligned} x(t) &= B_1 \cos(\sqrt{\lambda}t) + B_2 \sin(\sqrt{\lambda}t) \\ &= \sqrt{B_1^2 + B_2^2}\Big(\frac{B_1}{\sqrt{B_1^2 + B_2^2}} \cos(\sqrt{\lambda}t) + \frac{B_2}{\sqrt{B_1^2 + B_2^2}} \sin(\sqrt{\lambda}t)\Big). \end{aligned}$$

Suppose we denote $B := \sqrt{B_1^2 + B_2^2}$, $\sin(\theta) := \frac{B_1}{B}$ and $\cos(\theta) := \frac{B_2}{B}$, then using the trignomertic formula we get

$$x(t) = B \sin(\theta + \sqrt{\lambda}t),$$

We know that $x(0) \in \big(\frac{1}{1+\lambda}, 1\big)$, $\dot{x}(0) < 0$ and

$$x(t_0) = \frac{1}{1+\lambda} = B \sin(\theta + \sqrt{\lambda}t_0).$$

Therefore, the value of B in terms of $x(t_0)$ and $\dot{x}(t_0)$ is

$$B = \sqrt{\big(x(t_0)\big)^2 + \frac{1}{\lambda}\big(\dot{x}(t_0)\big)^2}, \tag{6.2.11}$$

(see Appendix A.2 for details). Since $x(t_0) = \frac{1}{1+\lambda}$ and $\dot{x}(t_0)$ is strickly less than zero, thus $B > \frac{1}{1+\lambda}$. Since $\sin^2(\theta + \sqrt{\lambda}t_0) + \cos^2(\theta + \sqrt{\lambda}t_0) = 1$, so, $\cos(\theta + \sqrt{\lambda}t_0) = -\sqrt{1 - \frac{1}{B^2(1+\lambda)^2}}$ and therefore

$$\dot{x}(t_0) = -\sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}.$$

Hence the solution $x(t)$ is decreasing for $x(t_0) = \frac{1}{1+\lambda}$ and $\dot{x}(t_0) = -\sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}$ and eventually reach $-\frac{1}{1+\lambda}$ at some time $\hat{t}$. The solution $x(\hat{t})$ at time $\hat{t} < t$ in this region from (6.2.13) is given by

$$x(\hat{t}) = C_1 e^{\hat{t}} + C_2 e^{-\hat{t}} - 1.$$

The solution in terms of the values of constants C_1 and C_2 (see Appendix A.3 for details) can be written as follows.

$$x(t) = \frac{1}{2}\Bigg(\frac{\lambda}{1+\lambda} - \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}\Bigg)e^{t-\hat{t}} + \frac{1}{2}\Bigg(\frac{\lambda}{1+\lambda} + \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}\Bigg)e^{\hat{t}-t} - 1$$

Note that we are interested in the case when the solution $x(t)$ decreases for short time and then turn around and increase. Since $B > \frac{1}{1+\lambda}$, it is obvious that $\frac{\lambda}{1+\lambda} + \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)} > 0$ but $\frac{\lambda}{1+\lambda} - \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}$ could be positive or negative. To show that the solution turns around after some time, it is sufficient to show that $\dot{x}(t) < 0$ if

$$e^{-2(t-\hat{t})} > \frac{\frac{\lambda}{1+\lambda} - \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}}{\frac{\lambda}{1+\lambda} + \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)}}. \tag{6.2.12}$$

holds for short time.

Let us suppose $\frac{\lambda}{1+\lambda} - \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)} > 0$. By replacing the values of $x(t_0)$ and $\dot{x}(t_0)$ in equation (A.2) for $\alpha < 0$ and $\beta < 0$ we obtained

$$B^2 = \frac{1}{(1+\lambda)^2} + \frac{1}{\lambda}\big(\alpha e^{t_0} - \beta e^{-t_0}\big)^2.$$

Since $\dot{x}(t_0) < 0$, then $\frac{\lambda}{1+\lambda} - \sqrt{\lambda\Big(B^2 - \frac{1}{(1+\lambda)^2}\Big)} > 0 \implies -\dot{x}(t_0) < \frac{\lambda}{1+\lambda}$. Hence,

$$\alpha e^{t_0} - \beta e^{-t_0} > -\frac{\lambda}{1+\lambda}.$$

Substituting value of t_0 and simplification gives that $\alpha\beta > 0$ which is true for $\alpha < 0$ and $\beta < 0$. Therefore, (6.2.12) holds for $\frac{\lambda}{1+\lambda} - \sqrt{\lambda\left(B^2 - \frac{1}{(1+\lambda)^2}\right)} > 0$ for small time $t > 0$, that is, the solution $x(t)$ is decreasing for short time and then start increasing. This indicates that the solution $x(t)$ turns around after some time and moves in the opposite direction of closest point 1 and reach $x(\bar{t}) = -\frac{1}{1+\lambda}$ at time $\bar{t}$. Then, we can argue as before for the modified region that the solution $x(t)$ increases and thus continues its oscillation between 1 and -1.

Case (ii): If $|x(0)| > 1$ and assume $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) < 0$, then the solution $x(t) \to \infty$ as $t \to \infty$.

Let $x(0) > 1 \iff x(0) - 1 > 0$, and $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) = -\frac{1}{2}\epsilon^2$, for some $\epsilon > 0$. Then, from the first formula of (6.2.9),

$$\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) = -\frac{1}{2}\epsilon^2 \implies \dot{x}(0) = \pm\sqrt{(x(0)-1)^2 - \epsilon^2}.$$

If $\dot{x}(0) = 0$, then $-F_\lambda(x(0)) = -\frac{1}{2}\epsilon^2 \implies x(0) = 1 + \epsilon$. Thus from the first formula of (6.2.13), the solution $x(t) \to \infty$ as $t \to \infty$.

If $\dot{x}(0) = \sqrt{(x(0)-1)^2 - \epsilon^2}$, then from the first formula of (6.2.13), $\dot{x}(0) + x(0) - 1 > 0$, and so the solution $x(t) \to \infty$ as $t \to \infty$.

If $\dot{x}(0) = -\sqrt{(x(0)-1)^2 - \epsilon^2}$, then from the first formula of (6.2.13),

$$\dot{x}(0) + x(0) - 1 > 0 \iff (x(0)-1)^2 > (x(0)-1)^2 - \epsilon^2 \geq 0.$$

So the solution $x(t) \to \infty$ as $t \to \infty$. Analagously, from the third formula of (6.2.13), the solution $x(t) \to -\infty$ as $t \to \infty$. Further cases we be will discuss in future.

Example 6.2.2. Let us consider the lower transform function $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ for the squared distance function to finite set $K = \{(-1,0),(1,0)\}$ for $\lambda > 0$.

$$C^l_\lambda \mathrm{dist}^2\big((x,y),K\big) = \begin{cases} (x-1)^2 + y^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda|x|^2 + y^2 & |x| \leq \frac{1}{1+\lambda} \\ (x+1)^2 + y^2 & x < -\frac{1}{1+\lambda} \end{cases}$$

The solution $x(t)$ and $y(t)$ in terms of initial conditions $x(0), y(0)$ and $\dot{x}(0), \dot{y}(0)$ are

$$x(t) = \begin{cases} \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^t + \frac{1}{2}\big(-\dot{x}(0) + x(0) - 1\big)e^{-t} + 1 & x(0) > \frac{1}{1+\lambda} \\ x(0)\cos(\sqrt{\lambda}t) + \frac{1}{\sqrt{\lambda}}\dot{x}(0)\sin(\sqrt{\lambda}t) & |x(0)| \leq \frac{1}{1+\lambda} \\ \frac{1}{2}\big(\dot{x}(0) + x(0) + 1\big)e^t + \big(-\dot{x}(0) + x(0) + 1\big)e^{-t} - 1 & x(0) < -\frac{1}{1+\lambda} \end{cases} . \quad (6.2.13)$$

$$y(t) = \frac{1}{2}\big(\dot{y}(0) + y(0)\big)e^t + \frac{1}{2}\big(-\dot{y}(0) + y(0)\big)e^{-t}$$

The behvaiour of the solution $x(t)$ has been described in previous example and here we consider $y(t)$ and provide sufficient conditions on $\dot{y}(0)$ for large $t > 0$ such that $y(t)$ is approaching 0, $+\infty$ or $-\infty$. These three cases of this solution $y(t)$ are given below.

Suppose that $\frac{1}{2}y(0) + \frac{1}{2}\dot{y}(0) = 0$. This implies that $\dot{y}(0) = -y(0)$. It is obvious that the terms $\frac{1}{2}\big(\dot{y}(0) + y(0)\big)e^t = 0$ and $\frac{1}{2}\big(-\dot{y}(0) + y(0)\big)e^{-t} \to 0$ as $t \to \infty$. Therefore, the solution $y(t) \to 0$ as $t \to \infty$.

Suppose that $\frac{1}{2}y(0) + \frac{1}{2}\dot{y}(0) > 0$. This implies that $\dot{y}(0) > -y(0)$. Since the terms $\frac{1}{2}\big(\dot{y}(0) + y(0)\big)e^t \to \infty$ and $\frac{1}{2}\big(-\dot{y}(0) + y(0)\big)e^{-t} \to 0$ as $t \to \infty$. Therefore, the solution $y(t) \to \infty$ when $t \to \infty$.

Suppose that $\frac{1}{2}y(0) + \frac{1}{2}\dot{y}(0) < 0$. This implies that $\dot{y}(0) < -y(0)$. Obviously the terms $\frac{1}{2}\big(\dot{y}(0) + y(0)\big)e^t \to -\infty$ and $\frac{1}{2}\big(-\dot{y}(0) + y(0)\big)e^{-t} \to 0$ as $t \to \infty$. Therefore, the solution $y(t) \to -\infty$ when $t \to \infty$

6.3 Qualitative properties of the dynamical system in $\mathbb{R}$ and the limits of solutions

In this section, we give a qualitative analysis of the example based on the lower transform $\frac{1}{2}C_\lambda^l \mathrm{dist}^2(x, K)$ of the squared distance function to the set $K = \{-1, 1\}$. This provides further information about the behaviour of the solution $x(t)$ of second order gradient system of the lower transform $\frac{1}{2}C_\lambda^l \mathrm{dist}^2(x, K)$. We have used this qualitative property in Section 2 of this chapter. In particular, this method calculates the exact value of the solution $x(t)$ at which it start oscillating. Let us denote half of the lower transform

$\frac{1}{2}C_\lambda^l \mathrm{dist}^2(x, K)$ by $F_\lambda(x)$ and the squared distance function $\frac{1}{2}\mathrm{dist}^2(x, K)$ by $F(x)$. Then the second order differential equation $\ddot{x}(t) = DF_\lambda(x(t))$ implies that,

$$\ddot{x}(t) \cdot \dot{x}(t) = DF_\lambda(x(t)) \cdot \dot{x}(t)$$
$$\implies \frac{1}{2}\frac{d}{dt}|\dot{x}(t)|^2 = \frac{d}{dt}F_\lambda(x(t)) \implies \frac{d}{dt}\Big(\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda(x(t))\Big) = 0.$$

Therefore, we have the following conservation law

$$\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda(x(t)) = \frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)), \tag{6.3.14}$$

and similarly, we have

$$\frac{1}{2}|\dot{x}(t)|^2 - F(x(t)) = \frac{1}{2}|\dot{x}(0)|^2 - F(x(0)). \tag{6.3.15}$$

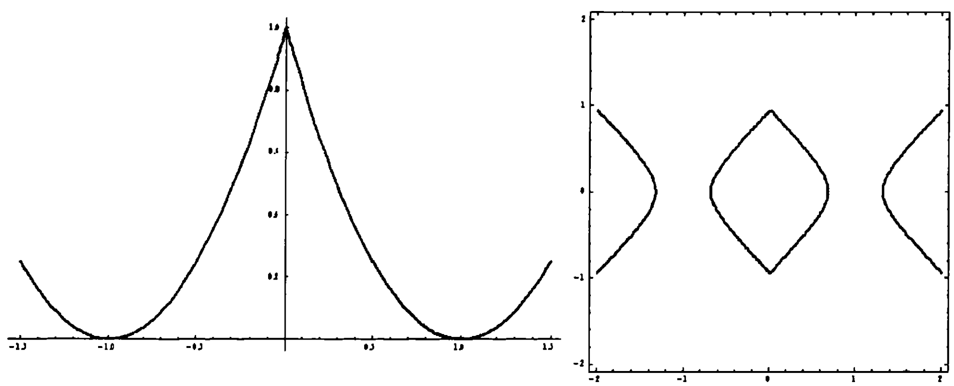

Figure 6.1: From left to right, the squared distance function $F(x) := \frac{1}{2}\mathrm{dist}^2(x, K)$ and the level set $\frac{1}{2}|\dot{x}(t)|^2 - F(x(t)) = -0.1$.

If the initial value $-1 < x(0) < 1$ and assume $\frac{1}{2}|\dot{x}(0)|^2 - F(x(0)) < 0$, then we note from Figure 6.1, that $\big(x(t), \dot{x}(t)\big)$ will stay on the closed curve, however, when the solution $x(t)$ approaches 0, the system becomes singular and there is no natural extension of solutions. The unbounded branches of the level set in Figure 6.1 and 6.2 remain same.

Case (i): If we take initial value $-1 < x(0) < 1$ and assume $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) < 0$, then we can see from Figure 6.2, that $\big(x(t), \dot{x}(t)\big)$ will stay on the closed curve and the unique solution $x(t)$ exists for all time t. Hence this shows that the quantity $|\frac{1}{2}\dot{x}(t)|^2 - F_\lambda(x(t))$

will remain the same for all time and thus the solution has a periodic behaviour for all time t provided $x(0) \in (-1, 1)$.

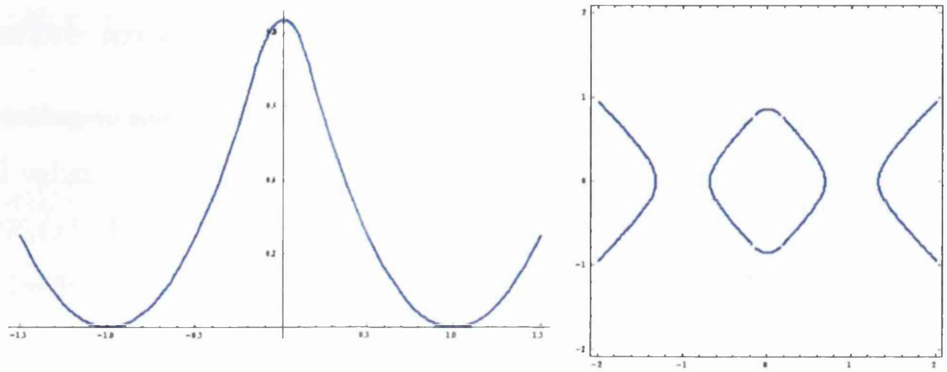

Figure 6.2: From left to right, the plot of the lower transform $F_\lambda(x) := \frac{1}{2}C^l_\lambda \mathrm{dist}^2(x, K)$ and the level set $\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda(x(t)) = -0.1$ for $\lambda = 5$.

Case (ii): If $|x(0)| > 1$ and assume $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) < 0$, then $\big(x(t), \dot{x}(t)\big)$ lies on one of the unbounded branches of the level set in Figure 6.2. Hence from Figure 6.2, when the initial value $x(0) > 1$, the solution $x(t) \to +\infty$ as $t \to \infty$ and when the initial value $x(0) < -1$, the solution $x(t) \to -\infty$ as $t \to \infty$.

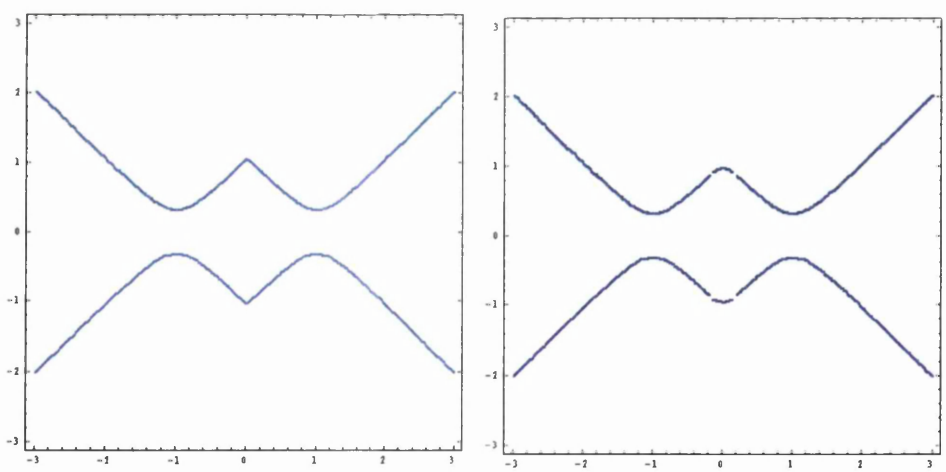

Figure 6.3: The plot of level set $\frac{1}{2}|\dot{x}(t)|^2 - F(x(t)) = 0.1$, where $F(x) := \frac{1}{2}\mathrm{dist}^2(x, K)$ to the left and plot of level set $\frac{1}{2}|\dot{x}(t)|^2 - F_\lambda(x(t)) = 0.1$, where $F_\lambda(x) := \frac{1}{2}C^l_\lambda \mathrm{dist}^2(x, K)$ to the right, for $\lambda = 5$.

Case (iii): In this case, we take $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) > 0$ as shown in Figure 6.3, and will analyse such cases in future.

Qualitative analysis of dynamical system when $\lambda \to \infty$

It is interesting to know what will happen to the modified system when $\lambda \to \infty$. Suppose the initial value $-1 < x(0) < 1$ and let $\frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(x(0)) < 0$, where x_λ is the solution of $\ddot{x} = DF_\lambda(x)$. In this case $\big(x_\lambda(t), \dot{x}_\lambda(t)\big)$ stays on the closed curve as shown in Figure 6.2 and clearly, $|\dot{x}_\lambda(t)|^2$ is uniformly bounded, hence $x_\lambda(t)$ is uniformly Lipschitz. By Arzela-Ascoli Theorem [16, Theorem 2.5], there is a subsequence $\lambda_n \to +\infty$ such that $x_{\lambda_n} \to x_\infty$ uniformly in bounded sets. Since from (6.2.8), we have

$$DF_\lambda(x) = \begin{cases} x-1 & x > \frac{1}{1+\lambda} \\ -\lambda x & -\frac{1}{1+\lambda} \le x \le \frac{1}{1+\lambda} \\ x+1 & x < -\frac{1}{1+\lambda} \end{cases},$$

so $|DF_\lambda(x)| \le 1+|x|$ implies that $\dot{x}_\lambda(t)$ is Lipschitz in $\mathbb{R}$ as $x_\lambda(t)$ is bounded. Therefore, by Arzela-Ascoli Theorem [16, Theorem 2.5], there exists a subsequence λ_{n_j} of λ_n such that $\dot{x}_{\lambda_{n_j}} \to \dot{x}_\infty$ uniformly in bounded intervals. Since $F_\lambda(y) \to F(y)$ uniformly as $\lambda \to \infty$. If we write $X_\lambda(t) = \big(x_\lambda(t), \dot{x}_\lambda(t)\big)$, where $x_\lambda(t)$ solves (6.2.5), and let Γ_λ be the level set

$$\frac{1}{2}|\dot{x}_\lambda(t)|^2 - F_\lambda(x(t)) = \frac{1}{2}|\dot{x}(0)|^2 - F_\lambda(0)$$

and let Γ be the level set

$$\frac{1}{2}|y|^2 - F(x) = \frac{1}{2}|\dot{x}(0)|^2 - F(0),$$

where $F(x) = \frac{1}{2}\text{dist}^2(x, K)$ and $F_\lambda(x) = \frac{1}{2}C^l_\lambda \text{dist}^2(x, K)$, then

$$\sup_{t \ge 0} \text{dist}\big(X_\lambda(t), \Gamma\big) \to 0 \quad \text{as} \quad \lambda \to \infty,$$

because $\Gamma_\lambda \to \Gamma$ as $\lambda \to \infty$ in the sense of Huasdorff distance, and we have as $\lambda_{n_j} \to \infty$,

$$\frac{1}{2}|\dot{x}_\infty(t)|^2 - F(x_\infty(t)) = \frac{1}{2}|\dot{x}(0)|^2 - F(x(0)).$$

So we may consider x_∞ as a general solution of the original system.

6.4 The behaviour of the solutions using first order smoothed gradient systems

We discuss briefly the long-time behaviour of the solutions of first order smoothed gradient systems for large $\lambda > 0$. Let us consider the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to the set $K = \{-1, 1\} \subset \mathbb{R}$. The explicit formula of the lower transform for $\lambda > 0$ is

$$C^l_\lambda \mathrm{dist}^2(x, K) = \begin{cases} |x-1|^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda|x|^2 & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ |x+1|^2 & x < -\frac{1}{1+\lambda}. \end{cases}$$

The solution of this first order gradient system in terms of initial condition $x(0)$ is of the form

$$x(t) = \begin{cases} \big(x(0)-1\big)e^t + 1 & x > \frac{1}{1+\lambda} \\ x(0)e^{-\lambda t} & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ \big(x(0)+1\big)e^t - 1 & x < -\frac{1}{1+\lambda} \end{cases} . \tag{6.4.16}$$

We investigate the behaviour of the solution $x(t)$ when the initial condition $x(0)$ is between -1 and 1 (i.e., inside the convex hull of set K). In fact, we will show that for $x(0)$ between -1 and 1, the solution $x(t) \to 0$ as $t \to \infty$ and for $|x(0)| > 1$, the solution $x(t)$ tends to ∞ as $t \to \infty$. In other words, the gradient flow of the lower transform $C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to $K = \{-1, 1\}$ converges to 0 as $t \to \infty$.

First suppose that the initial condition $x(0) \in \big(-\frac{1}{1+\lambda}, \frac{1}{1+\lambda}\big)$, then from second formula of (6.4.16), we have

$$x(t) = x(0)e^{-\lambda t} \to 0 \quad \text{as } t \to \infty,$$

since the term $e^{-2\lambda t} \to 0$ as $t \to \infty$. On the other hand, the gradient in this case,

$$\frac{1}{2} D C^l_\lambda \mathrm{dist}^2(x, K) = -\lambda x,$$

implies that the gradient flow points directily in the direction towards 0.

Now, let us suppose that the initial condition $x(0) \in \left(\frac{1}{1+\lambda}, 1\right)$, then from first formula of (6.4.16), we have

$$x(t) = \big(x(0) - 1\big)e^t + 1,$$

The solution $x(t)$ is decreasing. Let for some time t_0 such that $0 \le t < t_0$,

$$x(t_0) = \frac{1}{1+\lambda} = \big(x(0) - 1\big)e^{t_0} + 1.$$

Solving this for t_o gives

$$t_0 = \ln \frac{-\lambda}{(1+\lambda)(x(0)-1)}.$$

Also from second formula of (6.4.16) for time t_0

$$x(t_0) = Be^{-\lambda t_0} = \frac{1}{1+\lambda},$$

which gives $B = \frac{1}{1+\lambda}\left(\frac{-\lambda}{(1+\lambda)(x(0)-1)}\right)^{\lambda}$. Therefore, from first formula of (6.4.16), the solution

$$x(t) = \frac{1}{1+\lambda}\Big(\frac{-\lambda}{(1+\lambda)(x(0)-1)}\Big)^{\lambda} e^{-\lambda t} \to 0 \ \text{ as } t \to \infty,$$

because the term $e^{-\lambda t} \to 0$ as $t \to \infty$. Analagously, when $x(0) \in \left(-1, -\frac{1}{1+\lambda}\right)$, the solution $x(t) \to 0$, as $t \to \infty$.

Finally we suppose that the initial condition $|x(0)| > 1$. If $x(0) > 1 \iff x(0)-1 > 0$, then the solution $x(t)$ from first forumula of (6.4.16)

$$x(t) = \big(x(0) - 1\big)e^t + 1 \to \infty \ \text{ as } \ t \to \infty,$$

since the term $e^t \to \infty$ as $t \to \infty$. If $x(0) < -1 \iff x(0) + 1 < 0$, then the solution $x(t)$ from third forumula of (6.4.16)

$$x(t) = \big(x(0) + 1\big)e^t - 1 \to -\infty \ \text{ as } \ t \to \infty,$$

since the term $-e^t \to -\infty$ as $t \to \infty$.

Hence we have shown that if the initial condition $x(0)$ is in the interior of $C(K)$, then the solution $x(t) \to 0$ as $t \to \infty$ and if the initial condition lies outside $C(K)$, then the solution either tends to $+\infty$ or $-\infty$ as $t \to \infty$. Therefore, we note that inside the convex hull $C(K)$, the gradient flow converges to 0 as $t \to \infty$ and diverges otherwise.

Appendices

Appendix A

Explicit Calculations of Constants in Chapter 6

We include some of the computations of Example (6.2.1) in Chapter 6. We will calculate time t_0, constants B, B_1, B_2, C_1 and C_2.

A.1 Calculation of time t_0

We calculate the value of time t_0 for which the solution $x(t)$ approaches $\frac{1}{1+\lambda}$, that is, $x(t_0) = \frac{1}{1+\lambda}$. We know from formula (6.2.13) that

$$x(t_0) = \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^{t_0} + \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big)e^{-t_0} + 1. \tag{A.1.1}$$

Therefore, we have

$$\begin{aligned} & \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^{t_0} + \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big)e^{-t_0} + 1 = \frac{1}{1+\lambda} \\ \Longrightarrow \quad & \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^{2t_0} + \frac{\lambda}{1+\lambda}e^{t_0} + \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big) = 0 \end{aligned}$$

Solving this quadratic equation in $\gamma := e^{t_0}$ where $\gamma > 0$ for t_0 we get

$$\frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)\gamma^2 + \frac{\lambda}{1+\lambda}\gamma + \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big) = 0.$$

Hence,

$$e^{t_0} = \frac{-\frac{\lambda}{1+\lambda} \pm \sqrt{\frac{\lambda^2}{(1+\lambda)^2} - \big(\dot{x}(0) + x(0) - 1\big)\big(- \dot{x}(0) + x(0) - 1\big)}}{\dot{x}(0) + x(0) - 1} \tag{A.1.2}$$

It is assumed that $\frac{1}{1+\lambda} < x(0) < 1$ and $x(0)-1 < \dot{x}(0) < 0$, and let $\alpha := \frac{1}{2}\big(\dot{x}(0)+x(0)-1\big)$ and $\beta := \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big)$. Then for $\alpha < 0$ and $\beta < 0$ we take

$$t_0 = \ln \frac{-\frac{\lambda}{1+\lambda} - \sqrt{\frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta}}{2\alpha}$$

This is because for positive t_0,

$$\begin{aligned}
e^{t_0} > 1 \quad &\iff \quad \frac{-\frac{\lambda}{1+\lambda} \pm \sqrt{\frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta}}{2\alpha} > 1 \\
&\iff \quad -\frac{\lambda}{1+\lambda} \pm \sqrt{\frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta} < 2\alpha \qquad \text{(since } \alpha < 0 \text{)} \\
&\iff \quad \pm\sqrt{\frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta} < 2\alpha + \frac{\lambda}{1+\lambda}
\end{aligned}$$

Now consider

$$\begin{aligned}
&\sqrt{\frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta} < 2\alpha + \frac{\lambda}{1+\lambda} \\
\iff \quad & \frac{\lambda^2}{(1+\lambda)^2} - 4\alpha\beta < \frac{\lambda^2}{(1+\lambda)^2} + 4\alpha^2 + 4\frac{\alpha\lambda}{1+\lambda} \\
\iff \quad & 0 < \beta + \alpha + \frac{\lambda}{1+\lambda} \qquad \text{since } \alpha < 0
\end{aligned}$$

By substituting the values of α and β we obtained $x(0) < \frac{1}{1+\lambda}$, which contradicts the assumption $\frac{1}{1+\lambda} < x(0) < 1$. Therefore, in order to take $e^{t_0} > 1$, we choose the negative sign in (A.1.2) to get t_0.

A.2 Calculation of $B := \sqrt{B_1^2 + B_2^2}$

First we calculate B_1 and B_2 when

$$B_1 \cos(\sqrt{\lambda} t_0) + B_2 \sin(\sqrt{\lambda} t_0) = \alpha e^{t_0} + \beta e^{-t_0} + 1 \tag{A.2.3}$$

and its derivative

$$-\sqrt{\lambda}B_1\sin(\sqrt{\lambda}t_0)+\sqrt{\lambda}B_2\cos(\sqrt{\lambda}t_0)=\alpha e^{t_0}-\beta e^{-t_0} \qquad \text{(A.2.4)}$$

where $\alpha=\frac{1}{2}\big(\dot{x}(0)+x(0)-1\big)<0$ and $\beta=\frac{1}{2}\big(-\dot{x}(0)+x(0)-1\big)<0$. Therefore, solving equation (A.2.3) and (A.2.4) for B_1 and B_2 give

$$B_1=\alpha\big[\sqrt{\lambda}\cos(\sqrt{\lambda}t_0)-\sin(\sqrt{\lambda}t_0)\big]e^{t_0}+\beta\big[\sqrt{\lambda}\cos(\sqrt{\lambda}t_0)+\sin(\sqrt{\lambda}t_0)\big]e^{-t_0}+\cos(\sqrt{\lambda}t_0)$$

$$B_2=\alpha\big[\sqrt{\lambda}\sin(\sqrt{\lambda}t_0)+\cos(\sqrt{\lambda}t_0)\big]e^{t_0}+\beta\big[\sqrt{\lambda}\sin(\sqrt{\lambda}t_0)-\cos(\sqrt{\lambda}t_0)\big]e^{-t_0}+\sin(\sqrt{\lambda}t_0).$$

By taking the square of B_1 and B_2 and adding them give,

$$B:=\sqrt{B_1^2+B_2^2}=\sqrt{x^2(t_0)+\frac{1}{\lambda}\dot{x}^2(t_0)}$$

A.3 Calculation of C_1 and C_2

To calculate values of constants C_1 and C_2, we know the solution $x(\hat{t})$ at time $\hat{t}$ in the region where $-1<x<-\frac{1}{1+\lambda}$ from formula (6.2.13) is

$$x(\hat{t})=C_1e^{\hat{t}}+C_2e^{-\hat{t}}-1=-\frac{1}{1+\lambda}$$

and its derivative is

$$\dot{x}(\hat{t})=C_1e^{\hat{t}}-C_2e^{-\hat{t}}.$$

Since we know that $x(\hat{t})=B\sin(\theta+\sqrt{\lambda}\hat{t})$ and $\cos(\theta+\sqrt{\lambda}\hat{t})=\sqrt{1-\frac{1}{B^2(1+\lambda)^2}}$. Therefore,

$$\dot{x}(\hat{t})=-B\sqrt{\lambda}\cos(\theta+\sqrt{\lambda}\hat{t})=-B\sqrt{\lambda}\sqrt{1-\frac{1}{B^2(1+\lambda)^2}}.$$

Thus adding up the following equations,

$$\begin{aligned} C_1e^{\hat{t}}+C_2e^{-\hat{t}}-1 &= -\frac{1}{1+\lambda} \\ C_1e^{\hat{t}}-C_2e^{-\hat{t}} &= -B\sqrt{\lambda}\sqrt{1-\frac{1}{B^2(1+\lambda)^2}} \end{aligned}$$

give us

$$C_1=\frac{1}{2}\left(\frac{\lambda}{1+\lambda}-\sqrt{\lambda\Big(B^2-\frac{1}{(1+\lambda)^2}\Big)}\right)e^{-\hat{t}}$$

and

$$C_2 = \frac{1}{2}\left(\frac{\lambda}{1+\lambda} + \sqrt{\lambda\left(B^2 - \frac{1}{(1+\lambda)^2}\right)}\right)e^{\hat{t}}.$$

Appendix B

More Details on the Solutions of the Dynamical Systems Discussed in the Thesis

We gather the details of computations regarding dynamical systems and keep them here for references. We calculate the first and second order gradient systems of the lower transform of the squared distance function for specific examples and their corresponding solutions in $\mathbb{R}$,$\mathbb{R}^2$ and $\mathbb{R}^3$. In general, for $x \in \mathbb{R}^n$ first order gradient system of the lower transform $h(x) = C^l_\lambda \mathrm{dist}^2(x, K)$ of the squared distance function to a finite set $K \subset \mathbb{R}^n$ is

$$\begin{aligned} \dot{x}(t) &= Dh(x) \\ x(0) &= u_0, \end{aligned}$$

and the second order gradient system,

$$\begin{aligned} \ddot{x}(t) &= Dh(x) \\ x(0) &= u_0, \quad \dot{x}(0) = w_0. \end{aligned}$$

The explicit formulae for solutions of the first order gradient systems are calculated in first section and in the second section the explicit formulae for the solutions of the second order gradient systems in detail.

B.1 First-Order Gradient Systems and their Solutions

In this section we calculate the explicit formulae for first-order gradient systems of the lower transforms $\frac{1}{2}C_{\lambda}^{l}dist^2(x,K)$ of the squared distance function to finite sets $K \subset \mathbb{R}^n$ for $n = 1, 2, 3$ and their corresponding solutions. In particular, the solutions of the first order gradient systems of [21, Example 3],[21, Example 4], and Example (4.2.2), Example (4.2.3) and Example (4.2.4).

Explicit Formula for the solution of first order gradient system of the lower transform of the squared distance function to set $K = \{-1, 1\}$. The explicit formula for the lower transform $h(x) := C_{\lambda}^{l}dist^2(x,K)$ from [21, Example 3] is given by

$$h(x) = \begin{cases} |x-1|^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda|x|^2 & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ |x+1|^2 & x < -\frac{1}{1+\lambda} \end{cases}$$

We calculate first order gradient system of the lower transform $h(x)$

$$\dot{x}(t) = \begin{cases} 2(x-1) & x > \frac{1}{1+\lambda} \\ -2\lambda x & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ 2(x+1) & x < -\frac{1}{1+\lambda}, \end{cases}$$

and the explicit solutions for $\dot{x}(t) = 2x-2$, $\dot{x}(t) = -2\lambda x$ and $\dot{x}(t) = 2x+2$ separately. Let us consider the differential equation $\dot{x}(t) = 2x - 2$, then the solution $x(t)$ can be written as

$$x(t) = Ae^{2t} + 1, \qquad \text{where } A \text{ is a constant.}$$

The solution $x(t)$ in terms of initial condition $x(0) > \frac{1}{1+\lambda}$ is given by

$$x(t) = (x(0) - 1)e^{2t} + 1.$$

Now consider the differential equation $\dot{x}(t) = -2\lambda x$, then the solution $x(t)$ is of the form

$$x(t) = Be^{-2\lambda t}, \qquad \text{where } B \text{ is a constant.}$$

The solution $x(t)$ in terms of initial condition $-\frac{1}{1+\lambda} \leq x(0) \leq \frac{1}{1+\lambda}$ is given by $x(t) = x(0)e^{-2\lambda t}$.

Similarly, we consider the differential equation $\dot{x}(t) = 2x + 2$. Then the solution $x(t)$ is

$$x(t) = Ce^{2t} - 1, \qquad \text{where } C \text{ is a constant.}$$

The solution $x(t)$ in terms of initial condition $x(0) < -\frac{1}{1+\lambda}$ is of the form

$$x(t) = (x(0) + 1)e^{2t} - 1.$$

Hence the explicit formula for the solution of the first order gradient system of the lower transform $h(x)$ of squared distance function to finite set K is of the form

$$x(t) = \begin{cases} Ae^{2t} + 1 & x > \frac{1}{1+\lambda} \\ Be^{-2\lambda t} & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ Ce^{2t} - 1 & x < -\frac{1}{1+\lambda} \end{cases} \tag{B.1.1}$$

for A, B and C as constants and this forumula in terms of the initial condition $x(0)$ can be written as

$$x(t) = \begin{cases} (x(0) - 1)e^{2t} + 1 & x(0) > \frac{1}{1+\lambda} \\ x(0)e^{-2\lambda t} & -\frac{1}{1+\lambda} \leq x(0) \leq \frac{1}{1+\lambda} \\ (x(0) + 1)e^{2t} - 1 & x(0) < -\frac{1}{1+\lambda}. \end{cases} \tag{B.1.2}$$

Explicit Formula for the solution of first order gradient system of the lower transform of the squared distance function to set $K = \{(-1, 0), (1, 0), (0, 1)\}$. The lower transform $h(x, y) := C^l_\lambda \mathrm{dist}^2\big((x, y), K\big)$ from Example (4.2.2) is of the form

$$h(x,y) = \begin{cases} \frac{\lambda}{1+\lambda} - \lambda(x^2 + y^2) & |x| \geq y - \frac{1}{1+\lambda}, y \geq 0 \\ x^2 + (y-1)^2 & |x| \leq y - \frac{1}{1+\lambda}, y \geq \frac{1}{1+\lambda} \\ (x+1)^2 + y^2 & x \leq -\frac{1}{1+\lambda}, y \leq -x - \frac{1}{1+\lambda} \\ (x-1)^2 + y^2 & x \geq \frac{1}{1+\lambda}, y \leq x - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{2}\big(x + y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2 + y^2) & |x - y| < \frac{1}{1+\lambda}, x > -y + \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + \frac{1+\lambda}{2}\big(-x + y - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2 + y^2) & |x + y| < \frac{1}{1+\lambda}, x < y - \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} + y^2 - \lambda x^2 & |x| < \frac{1}{1+\lambda}, y < 0 \end{cases}$$

We compute the partial first order differential equations $\dot{x}(t) := h(x,y)$ and $\dot{y}(t) := h(x,y)$ with respect to x and y and find the solution to each partial first order differential equation. Note that most of the first order differential equations are similar to that of the previous example such as $\dot{x}(t) = 2(x+1)$, $\dot{x}(t) = 2(x-1)$ and $\dot{x}(t) = -2\lambda x$ and similarly for $\dot{y}(t)$. Hence, we only calculate the solutions of $\dot{x}(t)$ and $\dot{y}(t)$ for the following given regions. The first order partial differential equation $\dot{x}(t), \dot{y}(t)$ and their corresponding solution in terms of $x(0)$ and $y(0)$ are

$$\dot{x}(t) = \begin{cases} (1-\lambda)x + (1+\lambda)y - 1 & |x-y| < \frac{1}{1+\lambda}, x > -y + \frac{1}{1+\lambda} \\ -(1+3\lambda)x - (1+\lambda)y + 1 & |x+y| < \frac{1}{1+\lambda}, x < y - \frac{1}{1+\lambda} \end{cases}$$

$$x(t) = \begin{cases} \frac{1}{2}\big(x(0)+y(0)-1\big)e^{2t} + \frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t} + \frac{1}{2} & |x(0)-y(0)| < \frac{1}{1+\lambda}, x(0) > -y(0) + \frac{1}{1+} \\ \frac{1}{2}\big(x(0)-y(0)+1\big)e^{2t} + \frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t} - \frac{1}{2} & |x(0)+y(0)| < \frac{1}{1+\lambda}, x(0) < y(0) - \frac{1}{1+\lambda} \end{cases}$$

$$\dot{y}(t) = \begin{cases} (1+\lambda)x + (1-\lambda)y - 1 & |x-y| < \frac{1}{1+\lambda}, x > -y + \frac{1}{1+\lambda} \\ -(1+\lambda)x + (1-\lambda)y - 1 & |x+y| < \frac{1}{1+\lambda}, x < y - \frac{1}{1+\lambda} \end{cases}$$

$$y(t) = \begin{cases} \frac{1}{2}\big(x(0)+y(0)-1\big)e^{2t} - \frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t} + \frac{1}{2} & |x(0)-y(0)| < \frac{1}{1+\lambda}, x(0) > -y(0) + \frac{}{1} \\ -\frac{1}{2}\big(x(0)-y(0)+1\big)e^{2t} + \frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t} + \frac{1}{2} & |x(0)+y(0)| < \frac{1}{1+\lambda}, x(0) < y(0) - \frac{1}{1+} \end{cases}$$

Therefore, the solution for $\dot{x}(t)$ and $\dot{y}(t)$ in terms of $x(0)$ and $y(0)$ for all the regions is of the form

$$x(t) = \begin{cases} x(0)e^{-2\lambda t} & |x(0)| \geq y(0) - \frac{1}{1+\lambda}, y(0) \geq 0 \\ x(0)e^{2t} & |x(0)| \leq y(0) - \frac{1}{1+\lambda}, y(0) \geq \frac{1}{1+\lambda} \\ (x(0)+1)e^{2t} - 1 & x(0) \leq -\frac{1}{1+\lambda}, y(0) \leq -x(0) - \frac{1}{1+\lambda} \\ (x(0)-1)e^{2t} + 1 & x(0) \geq \frac{1}{1+\lambda}, y(0) \leq x(0) - \frac{1}{1+\lambda} \\ \frac{1}{2}\big(x(0)+y(0)-1\big)e^{2t} + \frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t} + \frac{1}{2} & |x(0)-y(0)| < \frac{1}{1+\lambda}, x(0) > -y(0) + \frac{1}{1+\lambda} \\ \frac{1}{2}\big(x(0)-y(0)+1\big)e^{2t} + \frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t} - \frac{1}{2} & |x(0)+y(0)| < \frac{1}{1+\lambda}, x(0) < y(0) - \frac{1}{1+\lambda} \\ x(0)e^{-2\lambda t} & |x(0)| < \frac{1}{1+\lambda}, y(0) < 0 \end{cases} \quad . \tag{B.1.3}$$

$$
y(t)=\begin{cases}
y(0)e^{-2\lambda t} & |x(0)|\geq y(0)-\frac{1}{1+\lambda}, y(0)\geq 0\\
(y(0)-1)e^{2t}+1 & |x(0)|\leq y(0)-\frac{1}{1+\lambda}, y(0)\geq \frac{1}{1+\lambda}\\
y(0)e^{2t} & x(0)\leq -\frac{1}{1+\lambda}, y(0)\leq -x(0)-\frac{1}{1+\lambda}\\
y(0)e^{2t} & x(0)\geq \frac{1}{1+\lambda}, y(0)\leq x(0)-\frac{1}{1+\lambda}\\
\frac{1}{2}\big(x(0)+y(0)-1\big)e^{2t}-\frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t}+\frac{1}{2} & |x(0)-y(0)|<\frac{1}{1+\lambda}, x(0)>-y(0)+\frac{1}{1+\lambda}\\
-\frac{1}{2}\big(x(0)-y(0)+1\big)e^{2t}+\frac{1}{2}\big(x(0)+y(0)\big)e^{-2\lambda t}+\frac{1}{2} & |x(0)+y(0)|<\frac{1}{1+\lambda}, x(0)<y(0)-\frac{1}{1+\lambda}\\
y(0)e^{2t} & |x(0)|<\frac{1}{1+\lambda}, y(0)<0.
\end{cases}
\tag{B.1.4}
$$

Explicit Formula for the solution of first order gradient system of the lower transform $h(x,y) := C^l_\lambda \mathbf{dist}^2\Big((x,y),K\Big)$ of the squared distance function to set $K=\big\{(0,1),\big(\frac{\sqrt{3}}{2},-\frac{1}{2}\big),\big(-\frac{\sqrt{3}}{2},-\frac{1}{2}\big)\big\}$. The lower transform from Example 4.2.3 is

$$
h(x,y)=\begin{cases}
\frac{\lambda}{1+\lambda}-\lambda(x^2+y^2) & |x|\geq \frac{1}{\sqrt{3}}\big(y-\frac{1}{1+\lambda}\big), y\geq \frac{-1}{2(1+\lambda)}\\
x^2+(y-1)^2 & |x|\leq \sqrt{3}\big(y-\frac{1}{1+\lambda}\big), y>\frac{1}{1+\lambda}\\
\big(x+\frac{\sqrt{3}}{2}\big)^2+\big(y+\frac{1}{2}\big)^2 & x<\frac{-\sqrt{3}}{2(1+\lambda)}, y\leq \frac{-1}{\sqrt{3}}x-\frac{1}{1+\lambda}\\
\big(x-\frac{\sqrt{3}}{2}\big)^2+\big(y+\frac{1}{2}\big)^2 & x>\frac{\sqrt{3}}{2(1+\lambda)}, y\leq \frac{1}{\sqrt{3}}x-\frac{1}{1+\lambda}\\
\frac{\lambda}{1+\lambda}+\frac{1+\lambda}{4}\big(\sqrt{3}x+y-\frac{1}{1+\lambda}\big)^2-\lambda(x^2+y^2) & \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)}<x<\frac{\sqrt{3}(1+y(1+\lambda)}{1+\lambda},\\
 & y<\frac{1}{\sqrt{3}}x+\frac{1}{1+\lambda}\\
\frac{\lambda}{1+\lambda}+\frac{1+\lambda}{4}\big(-\sqrt{3}x+y-\frac{1}{1+\lambda}\big)^2-\lambda(x^2+y^2) & \frac{\sqrt{3}(1+y(1+\lambda)}{1+\lambda}<x<\frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)},\\
 & y<-\frac{1}{\sqrt{3}}x+\frac{1}{1+\lambda}\\
\frac{\lambda}{1+\lambda}+(1+\lambda)\big(y+\frac{1}{2(1+\lambda)}\big)^2-\lambda(x^2+y^2) & |x|<\frac{\sqrt{3}}{2(1+\lambda)}, y<-\frac{1}{2(1+\lambda)}
\end{cases}
$$

We note that the first order partial differential equations for the first four regions and last region are similar to the first order partial differential equations of previous examples and so we will consider the following two regions in details. The first order partial differential equation $\dot{x}(t)$ of $h(x,y)$ for the considered regions is given by

$$
\dot{x}(t)=\begin{cases}
\frac{3-\lambda}{2}x+\frac{\sqrt{3}}{2}(1+\lambda)y-\frac{\sqrt{3}}{2(1+\lambda)}, & \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)}<x<\frac{\sqrt{3}(1+y(1+\lambda)}{1+\lambda}, y<\frac{1}{\sqrt{3}}x+\frac{1}{1+\lambda}\\
\frac{3-\lambda}{2}x-\frac{\sqrt{3}}{2}(1+\lambda)y+\frac{\sqrt{3}}{2(1+\lambda)}, & \frac{\sqrt{3}(1+y(1+\lambda)}{1+\lambda}<x<\frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)}, y<-\frac{1}{\sqrt{3}}x+\frac{1}{1+\lambda}
\end{cases}
$$

and $\dot{y}(t)$ for the considered regions is given by

$$\dot{y}(t) = \begin{cases} \frac{\sqrt{3}(1+\lambda)}{2}x + \frac{1-3\lambda}{2}y - \frac{1}{2}, & \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)} < x < \frac{\sqrt{3}(1+y(1+\lambda))}{1+\lambda}, y < \frac{1}{\sqrt{3}}x + \frac{1}{1+\lambda} \\ -\frac{\sqrt{3}(1+\lambda)}{2}x + \frac{1-3\lambda}{2}y - \frac{1}{2}, & \frac{\sqrt{3}(1+y(1+\lambda))}{1+\lambda} < x < \frac{1-y(1+\lambda)}{\sqrt{3}(1+\lambda)}, y < -\frac{1}{\sqrt{3}}x + \frac{1}{1+\lambda} \end{cases}$$

Hence, the following is the explicit formula for the solution $x(t)$ and $y(t)$ of first order differential equations in terms of $x(0)$ and $y(0)$.

$$x(t) = \begin{cases} x(0)e^{-2\lambda t} & |x(0)| \geq \frac{1}{\sqrt{3}}(y(0) - \frac{1}{1+\lambda}), y(0) \geq \frac{-}{2(1} \\ x(0)e^{2t} & |x(0)| \leq \sqrt{3}(y(0) - \frac{1}{1+\lambda}), y(0) > \frac{1}{1+} \\ (x(0) + \frac{\sqrt{3}}{2})e^{2t} - \frac{\sqrt{3}}{2} & x(0) < \frac{-\sqrt{3}}{2(1+\lambda)}, y(0) \leq \frac{-1}{\sqrt{3}}x(0) - \frac{1}{1+\lambda} \\ (x(0) - \frac{\sqrt{3}}{2})e^{2t} + \frac{\sqrt{3}}{2} & x(0) > \frac{\sqrt{3}}{2(1+\lambda)}, y(0) \leq \frac{1}{\sqrt{3}}x(0) - \frac{1}{1+\lambda} \\ \frac{\sqrt{3}}{4}\big(\frac{x(0)}{\sqrt{3}} - y(0)\big)e^{-2\lambda t} + \frac{3}{4}\big(x(0) + \frac{y(0)}{\sqrt{3}} - \frac{1}{\sqrt{3}}\big)e^{2t} + \frac{\sqrt{3}}{4} & \frac{-1}{\sqrt{3}}(y(0) - \frac{1}{1+\lambda}) < x(0) < \sqrt{3}y(0) + \\ & y(0) < \frac{1}{\sqrt{3}}x(0) + \frac{1}{1+\lambda} \\ \frac{\sqrt{3}}{2}\big(y(0) - \frac{x(0)}{\sqrt{3}} - 1\big)e^{-2\lambda t} + \frac{3}{2}\big(x(0) - \frac{y(0)}{\sqrt{3}} + \frac{1}{\sqrt{3}}\big)e^{2t} - \frac{\sqrt{3}}{4} & \sqrt{3}y(0) + \frac{\sqrt{3}}{1+\lambda} < x(0) < \frac{1}{\sqrt{3}}(y(0) - \\ & y(0) < -\frac{1}{\sqrt{3}}x(0) + \frac{1}{1+\lambda} \\ x(0)e^{-2\lambda t} & |x(0)| < \frac{\sqrt{3}}{2(1+\lambda)}, y(0) < -\frac{1}{2(1+\lambda)} \end{cases} \tag{B.1.5}$$

$$y(t) = \begin{cases} y(0)e^{-2\lambda t} & |x(0)| \geq \frac{1}{\sqrt{3}}(y(0) - \frac{1}{1+\lambda}), y(0) \geq \frac{-1}{2(1+.} \\ (y(0) - 1)e^{2t} + 1 & |x(0)| \leq \sqrt{3}(y(0) - \frac{1}{1+\lambda}), y(0) > \frac{1}{1+\lambda} \\ (y(0) + \frac{1}{2})e^{2t} - \frac{1}{2} & x(0) < \frac{-\sqrt{3}}{2(1+\lambda)}, y(0) \leq \frac{-1}{\sqrt{3}}x(0) - \frac{1}{1+\lambda} \\ (y(0) + \frac{1}{2})e^{2t} - \frac{1}{2} & x(0) > \frac{\sqrt{3}}{2(1+\lambda)}, y(0) \leq \frac{1}{\sqrt{3}}x(0) - \frac{1}{1+\lambda} \\ \frac{-3}{4}\big(\frac{x(0)}{\sqrt{3}} - y(0)\big)e^{-2\lambda t} + \frac{\sqrt{3}}{4}\big(x(0) + \frac{y(0)}{\sqrt{3}} - \frac{1}{\sqrt{3}}\big)e^{2t} + \frac{1}{4} & \frac{-1}{\sqrt{3}}(y(0) - \frac{1}{1+\lambda}) < x(0) < \sqrt{3}y(0) + \\ & y(0) < \frac{1}{\sqrt{3}}x(0) + \frac{1}{1+\lambda} \\ \frac{3}{2}\big(y(0) - \frac{x(0)}{\sqrt{3}} - 1\big)e^{-2\lambda t} + \frac{\sqrt{3}}{2}\big(x(0) - \frac{y(0)}{\sqrt{3}} + \frac{1}{\sqrt{3}}\big)e^{2t} + \frac{1}{4} & \sqrt{3}y(0) + \frac{\sqrt{3}}{1+\lambda} < x(0) < \frac{1}{\sqrt{3}}(y(0) - \frac{}{1} \\ & y(0) < -\frac{1}{\sqrt{3}}x(0) + \frac{1}{1+\lambda} \\ (y(0) + \frac{1}{2})e^{2t} - \frac{1}{2} & |x(0)| < \frac{\sqrt{3}}{2(1+\lambda)}, y(0) < -\frac{1}{2(1+\lambda)} \end{cases} \tag{B.1.6}$$

Explicit Formula for the solution of first order gradient system of the lower transform $h(x, y, z) := C^l_\lambda \mathbf{dist}^2\big((x, y, z), K\big)$ of the squared distance function to set $K = \big\{(1,1,1), (1,1,-1), (1,-1,1), (1,-1,-1), (-1,1,1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)\big\}$.

The lower transform from Example 4.2.4 is of the form

$$h = \begin{cases} \frac{3\lambda}{1+\lambda} - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \big(|x|-1\big)^2 + \big(|y|-1\big)^2 + \big(|z|-1\big)^2 & |x| > \frac{1}{1+\lambda}, |y| > \frac{1}{1+\lambda}, |z| > \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\big(|y| - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| > \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\big(|x| - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & |x| > \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\big(|z| - \frac{1}{1+\lambda}\big)^2 - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| > \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\Big(\big(|x| - \frac{1}{1+\lambda}\big)^2 + \big(|y| - \frac{1}{1+\lambda}\big)^2\Big) - \lambda(x^2+y^2+z^2) & |x| \ge \frac{1}{1+\lambda}, |y| \ge \frac{1}{1+\lambda}, |z| \le \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\Big(\big(|x| - \frac{1}{1+\lambda}\big)^2 + \big(|z| - \frac{1}{1+\lambda}\big)^2\Big) - \lambda(x^2+y^2+z^2) & |x| \ge \frac{1}{1+\lambda}, |y| \le \frac{1}{1+\lambda}, |z| \ge \frac{1}{1+\lambda} \\ \frac{3\lambda}{1+\lambda} + (1+\lambda)\Big(\big(|y| - \frac{1}{1+\lambda}\big)^2 + \big(|z| - \frac{1}{1+\lambda}\big)^2\Big) - \lambda(x^2+y^2+z^2) & |x| \le \frac{1}{1+\lambda}, |y| \ge \frac{1}{1+\lambda}, |z| \ge \frac{1}{1+\lambda} \end{cases}$$

Note that first order differential equations in this formula are reduced to similar first order differential equations in the previous examples, such as, $\dot{x}(t) = 2(x+1)$, $\dot{x}(t) = 2(x-1)$ and $\dot{x}(t) = -2\lambda$, and analogously for $\dot{y}(t)$ and $\dot{z}(t)$. Therefore, by comparing with the previous examples can get the explicit formula for the solution but here we give the formula for the following specific regions as these formulae are needed in application of lower transform of squared distance functions in sureface reconstructions. So we have

$$x(t) = \begin{cases} x(0)e^{-2\lambda t} & |x(0)| \le \frac{1}{1+\lambda}, |y(0)| \le \frac{1}{1+\lambda}, |z(0)| \le \frac{1}{1+\lambda} \\ \big(x(0)-1\big)e^{2t} + 1 & |x(0)| > \frac{1}{1+\lambda}, |y(0)| > \frac{1}{1+\lambda}, |z(0)| > \frac{1}{1+\lambda} \end{cases} \tag{B.1.7}$$

$$y(t) = \begin{cases} y(0)e^{-2\lambda t} & |x(0)| \le \frac{1}{1+\lambda}, |y(0)| \le \frac{1}{1+\lambda}, |z(0)| \le \frac{1}{1+\lambda} \\ \big(y(0)-1\big)e^{2t} + 1 & x(0) > \frac{1}{1+\lambda}, y(0) > \frac{1}{1+\lambda}, z(0) > \frac{1}{1+\lambda} \\ \big(y(0)-1\big)e^{2t} + 1 & x(0) > \frac{1}{1+\lambda}, y(0) > \frac{1}{1+\lambda}, |z(0)| \le \frac{1}{1+\lambda} \\ y(0)e^{-2\lambda t} & x(0) > \frac{1}{1+\lambda}, |y(0)| \le \frac{1}{1+\lambda}, |z(0)| \le \frac{1}{1+\lambda} \\ y(0)e^{-2\lambda t} & x(0) > \frac{1}{1+\lambda}, |y(0)| \le \frac{1}{1+\lambda}, z(0) > \frac{1}{1+\lambda} \end{cases} \tag{B.1.8}$$

$$z(t) = \begin{cases} z(0)e^{-2\lambda t} & |x(0)| \leq \frac{1}{1+\lambda}, |y(0)| \leq \frac{1}{1+\lambda}, |z(0)| \leq \frac{1}{1+\lambda} \\ \big(z(0)-1\big)e^{2t}+1 & x(0) > \frac{1}{1+\lambda}, y(0) > \frac{1}{1+\lambda}, z(0) > \frac{1}{1+\lambda} \\ z(0)e^{-2\lambda t} & x(0) > \frac{1}{1+\lambda}, y(0) > \frac{1}{1+\lambda}, |z(0)| \leq \frac{1}{1+\lambda} \\ z(0)e^{-2\lambda t} & x(0) > \frac{1}{1+\lambda}, |y(0)| \leq \frac{1}{1+\lambda}, |z(0)| \leq \frac{1}{1+\lambda} \\ \big(z(0)-1\big)e^{2t}+1 & x(0) > \frac{1}{1+\lambda}, |y(0)| \leq \frac{1}{1+\lambda}, z(0) > \frac{1}{1+\lambda} \end{cases} \tag{B.1.9}$$

B.2 Second-Order Gradient Systems and Solutions

This section contains the explicit formulae for second order gradient systems of the lower transforms for squared distance function to a finite sets and their solutions. In particular, we compute the solutions of second order gradient systems of previously calculated lower transform in example [21, Example (3)] and Example (4.2.1). These solutions are helpful in the application of the lower transform of squared distance function to a finite set to differential equations.

Explicit Formula for the solution of second order gradient system of the lower transform $h(x) := C^l_\lambda \mathbf{dist}^2(x, K)$ of the squared distance function to finite set $K = \{-1, 1\}$. The lower transform from Example [21, Example (3)] is of the form

$$h(x) = \begin{cases} |x-1|^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda|x|^2 & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ |x+1|^2 & x < -\frac{1}{1+\lambda} \end{cases}$$

The second order gradient system of $\frac{1}{2}h(x)$ is given by

$$\ddot{x}(t) = \begin{cases} x-1 & x > \frac{1}{1+\lambda} \\ -\lambda x & -\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda} \\ x+1 & x < -\frac{1}{1+\lambda} \end{cases} \tag{B.2.10}$$

We compute the solution to all the three second order differential equations separately. Therefore, let us consider the second order differential equation $\ddot{x}(t) = x - 1$ for which

$x > \frac{1}{1+\lambda}$. Then the solution $x(t)$ is,

$$x(t) = A_1 e^t + A_2 e^{-t} + 1, \qquad \text{where } A_1 \text{ and } A_2 \text{ are constants.}$$

We solve write $x(t)$ and $\dot{x}(t)$ in terms of $x(0)$ and $\dot{x}(0)$ and then solve for A_1 and A_2. Therefore, we get

$$\begin{aligned} A_1 &= \frac{1}{2}\dot{x}(0) + \frac{1}{2}x(0) - \frac{1}{2} \\ A_2 &= -\frac{1}{2}\dot{x}(0) + \frac{1}{2}x(0) - \frac{1}{2} \end{aligned}$$

and thus the solution $x(t)$ is then

$$x(t) = \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^t + \frac{1}{2}\big(- \dot{x}(0) + x(0) - 1\big)e^{-t} + 1.$$

Let us consider second order differential equation $\ddot{x}(t) = -\lambda x$ for which $-\frac{1}{1+\lambda} \leq x \leq \frac{1}{1+\lambda}$. Then the solution $x(t)$ is given by

$$x(t) = B_1 \cos(\sqrt{\lambda}t) + B_2 \sin(\sqrt{\lambda}t), \qquad \text{where } B_1 \text{ and } B_2 \text{ are constants.}$$

We write $x(t)$ and $\dot{x}(t)$ in terms of $x(0)$ and $\dot{x}(0)$ and then solve for B_1 and B_2 in terms of $x(0)$ and $\dot{x}(0)$. Thus we get $B_1 = x(0)$ and $B_2 = \frac{1}{\sqrt{2\lambda}}\dot{x}(0)$. Hence the solution $x(t)$ in terms of $x(0)$ and $\dot{x}(0)$ is as follows

$$x(t) = x(0)\cos(\sqrt{\lambda}t) + \frac{1}{\sqrt{\lambda}}\dot{x}(0)\sin(\sqrt{2\lambda}t).$$

Finally consider the second order differential equation $\ddot{x}(t) = x + 1$ for which $x < -\frac{1}{1+\lambda}$. Then the solution $x(t)$ is written as

$$x(t) = C_1 e^t + C_2 e^{-t} - 1, \qquad \text{where } C_1 \text{ and } C_2 \text{ are constants.}$$

We compute $x(t)$ and $\dot{x}(t)$ in terms of $x(0)$ and $\dot{x}(0)$ and solve for C_1 and C_2 that gives

$$\begin{aligned} C_1 &= \frac{1}{2}\dot{x}(0) + \frac{1}{2}x(0) + \frac{1}{2} \\ C_2 &= -\frac{1}{2}\dot{x}(0) + \frac{1}{2}x(0) + \frac{1}{2} \end{aligned}$$

Therefore, the solution is

$$x(t) = \frac{1}{2}\big(\dot{x}(0) + x(0) + 1\big)e^{t} + \frac{1}{2}\big(-\dot{x}(0) + x(0) + 1\big)e^{-t} - 1$$

In short, the explicit formula for the solution of the second order gradient system $\ddot{x}(t) = g'(x(t))$ for initial conditions $x(0)$ and $\dot{x}(0)$ is the following

$$x(t) = \begin{cases} \frac{1}{2}\big(\dot{x}(0) + x(0) - 1\big)e^{t} + \frac{1}{2}\big(-\dot{x}(0) + x(0) - 1\big)e^{-t} + 1 & x(0) > \frac{1}{1+\lambda} \\ x(0)\cos(\sqrt{\lambda}t) + \frac{1}{\sqrt{\lambda}}\dot{x}(0)\sin(\sqrt{2\lambda}t) & |x(0)| \leq \frac{1}{1+\lambda} \\ \frac{1}{2}\big(\dot{x}(0) + x(0) + 1\big)e^{t} + \frac{1}{2}\big(-\dot{x}(0) + x(0) + 1\big)e^{-t} - 1 & x(0) < -\frac{1}{1+\lambda} \end{cases} \tag{B.2.11}$$

Explicit Formula for the solution of second order gradient system of the lower transform $h(x,y) := C^{l}_{\lambda}\mathbf{dist}^2\big((x,y),K\big)$ of square distance function to finite set $K = \{(-1,0),(1,0)\}$. The lower transform from Example (4.2.1) by reordering x and y gives

$$h(x,y) = \begin{cases} (x-1)^2 + y^2 & x > \frac{1}{1+\lambda} \\ \frac{\lambda}{1+\lambda} - \lambda|x|^2 \; + y^2 & |x| \leq \frac{1}{1+\lambda} \\ (x+1)^2 + y^2 & x < -\frac{1}{1+\lambda} \end{cases}$$

Note that the lower transform $\frac{1}{2}h(x,y)$ has the same second order partial differential equations as in previous example with respect to x. The explicit formula for the solution $x(t)$ is already computed in previous example so we now compute the solution $y(t)$ for second order partial differential equation $\ddot{y}(t) = y$. Therefore, the solution is given by

$$y(t) = D_1e^{t} + D_2e^{-t}, \qquad \text{where } D_1 \text{ and } D_2 \text{ are constants.}$$

We compute $y(t)$ and $\dot{y}(t)$ in terms of $y(0)$ and $\dot{y}(0)$ and solve for D_1 and D_2,

$$\begin{aligned} D_1 &= \frac{1}{2}\dot{y}(0) + \frac{1}{2}y(0) \\ D_2 &= -\frac{1}{2}\dot{y}(0) + \frac{1}{2}y(0) \end{aligned}$$

Therefore, the solution in terms of $y(0)$ and $\dot{y}(0)$ is as follows

$$y(t) = \frac{1}{2}\big(\dot{y}(0) + y(0)\big)e^{t} + \frac{1}{2}\big(-\dot{y}(0) + y(0)\big)e^{-t}$$

Appendix C

MATLAB Implementation

We include the detailed MATLAB programming codes for the lower, upper and mixed transforms and implement them to plot these transforms for some specific example with the help of Antinio Orlando implementation for such transforms.

C.1 Lower Transforms

The first Matlab code is for computing one-dimensional Lower transform $C^l_\lambda \mathrm{dist}^2(x,K)$ of the squared distance function to set $K=\{1,-1\}$ for $\lambda=0.5$ where $x\in\mathbb{R}$ is between -2 and 2.

```
function LowerTransform1D
lambda = 0.5; x = -2:0.1:2;
f = min(abs(x-1).^2, abs(x+1).^2);
lg = f + lambda*(x.^2);
it=200; lf_in=lg; lf_out=lf_in;
for i=1:it
    for j=2:length(x)-1,
        lf_out(j)=min(lg(j),(lf_in(j-1) + lf_in(j+1))/2);
    end
```

```
    lf_out(1)=lf_in(1); lf_out(length(x))=lf_in(length(x)); lf_in=
        lf_out;
end
for i=1:length(x),
    lz(i)=lf_out(i)-lambda*x(i).^2;
end
plotyy(x,f,x,lz)
end
```

This second Matlab code is for computing two-dimensional Lower transform $C^l_\lambda \mathrm{dist}^2\big((x,y),K\big)$ to set $K=\Big\{(0,1),\big(\frac{\sqrt{3}}{2},-\frac{1}{2}\big),\big(-\frac{\sqrt{3}}{2},-\frac{1}{2}\big)\Big\}$ for $\lambda=0.5$ where $x,y\in\mathbb{R}$ and $(x,y)\in\mathbb{R}^2$ are both between -2 and 2.

```
function LowerTransform2D
lambda = 5; x = -1:0.03:1; y = -1:0.03:1;
[X,Y] = meshgrid(x,y);
f = min(min(X.^2+ (Y-1).^2,(X- sqrt(3)/2).^2 + (Y+ 1/2).^2),(X+
    sqrt(3)/2).^2 + (Y+ 1/2).^2);
lg = f + lambda*(X.^2 + Y.^2);
it=200; lf_in=lg; lf_out=lf_in;
for i=1:it
    for j=2:length(x)-1,
        for k=2:length(y)-1,
            lf_out(j,k)= min(lg(j,k),min(min((lf_in(j-1,k) +
                lf_in(j+1,k))/2,(lf_in(j,k+1) +lf_in(j,k-1))/2),
                min((lf_in(j-1,k-1) + lf_in(j+1,k+1))/2,(lf_in(j
                -1,k+1) + lf_in(j+1,k-1))/2)));
        end
    end
    lf_out(1)=lf_in(1); lf_out(length(x))=lf_in(length(x)); lf_in=
        lf_out;
```

```
end
for j=1:length(x),
    for k=1:length(y),
        Lz(j,k)=lf_out(j,k)-lambda*(X(j,k).^2+Y(j,k).^2);
    end
end
figure
subplot(1,2,1)
surf(X,Y,f)
subplot(1,2,2)
surf(X,Y,Lz)
end
```

C.2 Upper Transforms

The first Matlab code is for computing one-dimensional Upper transform of a given function, which in this code is the squared distance function to a finite set $K = \{1, -1\}$ for $\lambda = 0.5$ where $x \in \mathbb{R}$ is between -2 and 2.

```
function UpperTransform1D
lambda=0.7; x=-2:0.05:2;
f = min(abs(x-1), abs(x+1));
lg = -f + lambda*(x.^2);
it=200; lf_in=lg; lf_out=lf_in;
for i=1:it
    for j=2:length(x)-1,
        lf_out(j)=min(lg(j),(lf_in(j-1) + lf_in(j+1))/2);
    end
    lf_out(1)=lf_in(1);lf_out(length(x))=lf_in(length(x));lf_in=
      lf_out;
```

```
end
for i=1:length(x),
      lz(i)=lf_out(i)-lambda*x(i).^2;
      uz(i)=-lz(i);
end
plotyy(x,f,x,uz)
end
```

This second Matlab code is for computing two-dimensional Upper transform of maximum like function for $\lambda = 0.5$ where $x, y \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ are both between -2 and 2.

```
function UpperTransform2D
lambda=0.05; x=-2:0.05:2; y=-2:0.05:2;
[X,Y] = meshgrid(x,y);
f = max(abs(X),abs(Y));
g = lambda*(X.^2 + Y.^2)-f;
f_in=g;
f_out=f_in;
for m=1:100
      f_out=g;
      for j=2:length(x)-1,
            for k=2:length(y)-1,
                  f_out(j,k)=min(g(j,k),min(min((f_in(j-1,k) +f_in(j+1,k))/2,(f_in(j,k+1) +f_in(j,k-1))/2),min((f_in(j-1,k-1) +f_in(j+1,k+1))/2,(f_in(j-1,k+1) +f_in(j+1,k-1))/2)));
            end
      end
      f_in=f_out;
end
```

```
z=f;
for j=2:length(x)-1,
    for k=2:length(y)-1,
        z(j,k)=-lambda*(X(j).^2+Y(k).^2)-f_out(j,k);
    end
end
figure
subplot(1,2,1)
mesh(X,Y,f)
subplot(1,2,2)
mesh(X,Y,z)
view(0,0)
end
```

C.3 Mixed Transforms

The Matlab code in this section is computing one-dimensional Mixed transforms of a given function, which in this code is the squared distance function to set $K = \{1, -1\}$ for $\lambda = 1.5$ and $\alpha = 1.5$ where $x \in \mathbb{R}$ is between -2 and 2.

```
function MixTransform1D
lambda =1.5; alpha =1.5; x = -2:0.1:2;
f = min(abs(x-1),abs(x+1));
lg = f + lambda*(x.^2);
it=200;
lf_in=lg;
lf_out=lf_in;
for i=1:it
    for j=2:length(x)-1,
        lf_out(j)=min(lg(j),(lf_in(j-1) + lf_in(j+1))/2);
```

```
    end
    lf_out(1)=lf_in(1);
    lf_out(length(x))=lf_in(length(x));
    lf_in=lf_out;
end
for i=1:length(x),
    lz(i)=lf_out(i)-lambda*x(i).^2;
end
ug = -f + alpha*(x.^2);
uf_in=ug;
uf_out=uf_in;
for i=1:it
    for j=2:length(x)-1,
        uf_out(j)=min(ug(j),(uf_in(j-1) + uf_in(j+1))/2);
    end
    uf_out(1)=uf_in(1);
    uf_out(length(x))=uf_in(length(x));
    uf_in=uf_out;
end
for i=1:length(x),
    uz(i)=-uf_out(i)+alpha*x(i).^2;
end
figure
plotyy(x,lz,x,uz)
end
```

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